A Unified Approach to Structural Analysis and Design of Model Predictive Controllers

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***Abstract*—Designing linear MPC with pre-specified closed-loop characteristics for stability and robustness consideration as well as optimal time domain performance, is an interesting issue. In this paper, we develop a new enabling formulation, which can explicitly show existence and properties of the linear controller counterpart for transfer function-based MPC, known as Generalized Predictive Control. This development allows one to transform desired closed loop specifications to constraints on new-defined variables of the GPC optimization problem along with desired time domain performance-related design parameters. Input output constraints also can be transformed to constraints on these new variables. Fantastic results are illustrated by an ongoing example. It is a unified approach to answer some key questions in both theory and application such as analysis and design for desired performance, stability and robustness, controller matching, reference governor GPC, and design of model reference predictive control in data-driven control.**

***Keywords*: model predictive control, pole-placement predictive control, optimal control, robust control, controller matching.**

I. INTRODUCTION

MODEL predictive control (MPC) is the most applied advanced control scheme due to its attractive features. However, in contrast with other linear control techniques, closed-loop properties such as stability and robustness are usually not taken into account in MPC design [1]. Thus, the general question: "how MPC closed loop characteristics can be analysed and shaped, and how trade-off between stability and performance can be formulated and optimized?" has been raised from the early times of its development. It is well-known that every linear transfer function-based MPC (Generalized Predictive Control –GPC) can be transformed to a linear controller (general two-degree of freedom -RST- controller), and its characteristics can be analyzed by its counterpart instead [2]. So specific questions which arise are: 1) What relationships are there between GPC design parameters and counterpart RST controller terms? 2) For a desired closed loop characteristic polynomial, how design parameters of the counterpart GPC can be calculated or, should be selected? 3) When closed loop poles of a GPC set according to given requirements, how many degrees of freedom remain for time optimality performance consideration? 4) How constraints handling capabilities of the MPC can be considered along with closed loop characteristic requirements?

There are two different approaches to the problem. In the first approach, influences of the MPC design parameters on closed loop tracking transient response, including rise time and error as performance indicator and overshoot and settling time as robustness indicators, have been studied and some guidelines were suggested for tuning MPC (See [3] and review papers [4], [5], [6] and references there in).

The second approach, which is also followed in this paper, has been studying relationships between MPC design parameters and its counterpart linear controller terms. Mohtadi and Clarke [7] showed that both linear quadratic (LQ) and pole-placement control can be derived using GPC framework by choosing appropriate horizons and

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design polynomials. Fikar et.al. in [8] studied relationships between stable predictive control and pole placement. Landau and his colleagues [9] have mentioned that every RST controller is a one step ahead MPC, and an RST controller can be designed in the time domain using one step ahead MPC strategies, as already showed by Camacho and Bordons in [2] and references therein. Cairano and Bemporad in [1] addressed the following inverse problem: "how to select the performance index (in particular, the weighting matrices) of a linear MPC controller so that it behaves as a given favorite linear controller when the constraints are not activated". The problem also has been addressed as pole restricted GPC to find control weights to put closed loop poles in a desired - although shaped to be solvable - region in the z-plane [10]. Hartley and Maciejowski in [11] proposed a method using the observer-compensator realization of a more general class of stabilizing LTI output feedback controllers. Tran and colleagues in [12] proposed a method for finding weighting matrices in the cost function that will result in the GPC gain as required. Shah and Engel in [13] followed transfer function formulation of the problem to calculate GPC tuning parameters, but for some simplified cases.

However, although there have been many studies on the subject, some basic and important questions were ignored or answered incompletely. Among them are explicit limitations on, and relationships between, GPC design parameters and orders of RST controller terms, plant model, and closed loop characteristic polynomial, which should be considered appropriately in defining requirements. Also there are some degrees of freedom which should be recognized. In this paper we focus on two important issues. At first we reformulate GPC in a well-suited form which facilitates structural analysis of it and its counterpart RST controller in detail. Next we propose a new method to transform pole-placement designed RST controller to equality constraints on GPC controller gain. This derivation enables us to analyze various aspects of the pole-placement GPC. The proposed design algorithm can also consider constraint on input and output, which has not been addressed in the previous works. We also show that when constraints are consistent, the resulting controller is feasible and its stability is guaranteed. We also show that the resulting controller is a reference governor MPC; i.e., its counterpart controller comprises of an inner loop stabilizing controller and an outer reference governor (RG) controller [14], [15].

The remaining parts of this paper are as follows. In Section II we review the pole-placement design of RST controllers. In Section III a new formulation for GPC which is comparable to RST controller structure is introduced. In Section IV - the main part of the paper - the procedure of pole-placement design of GPC is developed. Some of its properties such as degrees of freedom for either of closed-loop pole-placement and time optimal performance and offset-free condition have been highlighted and proved. Detailed formulation for design of unconstrained linear GPC controller with pre-specified closed-loop requirements is derived in Section V. In Section VI the suggested method will be extended for inequality constraints on the plant's input and output. Looking at pole-placement GPC as a reference governor and its variants is discussed in Section VII. In section VIII several illustrative examples are used to clarify main results of the paper. Finally, in Section IX the work is concluded, and future works are addressed in Section X.

II. Pole-Placement Design of RST Controller

It is believed that all conventional linear controllers have equivalent RST counterparts. The pole-placement allows designing an RST controller for stable or unstable systems, without restriction upon the degrees of the numerator and denominator polynomials of the plant model and RHP zeros [16]. Figure 1 shows the structure of RST controller, where plant is described by

$A(z^{-1})y(t) = B(z^{-1})u(t)$

$A(z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_{n_A} z^{-n_A}$,

$B(z^{-1}) = b_1 z^{-1} + \cdots + b_{n_B} z^{-n_B}, b_i = 0, i = 1, \cdots, d$ (plant delay)

and the controller polynomials are defined as

$$\begin{aligned} R(z^{-1}) &= r_0 + r_1 z^{-1} + \cdots + r_{n_R} z^{-n_R}, \\ S(z^{-1}) &= 1 + s_1 z^{-1} + \cdots + s_{n_S} z^{-n_S}, \\ T(z^{-1}) &= t_0 + t_1 z^{-1} + \cdots + t_{n_T} z^{-n_T}. \end{aligned} \tag{1}$$

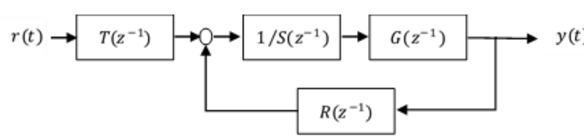

Fig. 1. Block diagram of the standard RST controller.

Desired closed-loop characteristic polynomial is in the Diophantine equation form

$$\begin{aligned} \bar{A}(z^{-1}) &= A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1}) \\ &= 1 + \bar{a}_1 z^{-1} + \cdots + \bar{a}_{n_{\bar{A}}} \end{aligned} \tag{2}$$

This equation can be solved when written in $Mx = c$ form as

$$\begin{array}{cc} \overbrace{\leftarrow -n_s - \rightarrow}^{} \quad\quad \leftarrow -n_R + 1 - \rightarrow & \\ \begin{bmatrix} 1 & \vdots & 0 & b_1 & \vdots & 0 \\ a_1 & \vdots & 0 & b_2 & \vdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n_A-1} & \vdots & 0 & b_{n_B-1} & \vdots & 0 \\ a_{n_A} & \vdots & 1 & b_{n_B} & \vdots & b_1 \\ 0 & \vdots & a_1 & 0 & \vdots & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & a_{n_A} & 0 & \vdots & b_{n_B} \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_{n_S} \\ r_0 \\ r_1 \\ \vdots \\ r_{n_R} \end{bmatrix} = \begin{bmatrix} \bar{a}_1 - a_1 \\ \bar{a}_2 - a_2 \\ \vdots \\ \bar{a}_{n_A} - a_{n_A} \\ \bar{a}_{n_A+1} \\ \vdots \\ \bar{a}_{n_{\bar{A}}} \end{bmatrix} & \end{array}$$

(3)

There are unique minimal solutions for $R(z^{-1})$ and $S(z^{-1})$ when polynomials $A(z^{-1})$ and $B(z^{-1})$ are coprime and the left hand matrix of (3) is full row rank [9], [17]. Thus, the number of independent equations in (3) is $q = n_A + n_B - 1$, the orders of $R(z^{-1})$, and $S(z^{-1})$ are $n_R = n_A - 1, n_S = n_B - 1$, and the order of desired $\bar{A}(z^{-1})$ should be

$$n_{\bar{A}} \leq n_A + n_B - 1. \tag{4}$$

Polynomials R and S can have pre-specified parts – a differentiator for example - to impose an integrator, or some robustness and/or closed loop performance requirements.

III. Generalized Predictive Control, A New Formulation

GPC is one of the well-developed MPC algorithms with good capabilities in control of various types of plants, and comparative features with transfer function-based controllers. The future outputs of system can be estimated using the corresponding difference equation $y(t+j) = \sum_{i=1}^{n_B} b_i u(t+j-i) - \sum_{i=1}^{n_A} a_i\, y(t+j-i)$. As $y(t+j-i)$ is not available for $j-i>0$, it is estimated based on past outputs recursively. Proceeding the same manipulation, one can arrange final result for $\hat{y}(t+1|t)$ up to $\hat{y}(t+d+P|t)$, (P is the prediction horizon), in the following matrix form

$$\begin{bmatrix} \hat{y}(t+1|t) \\ \hat{y}(t+2|t) \\ \vdots \\ \hat{y}(t+d+P|t) \end{bmatrix} =$$
$$\begin{bmatrix} b_1 & 0 & \dots & 0 \\ -a_1 b_1 + b_2 & b_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ & & \cdots & b_1 \end{bmatrix} \begin{bmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+P-1) \end{bmatrix} +$$
$$\begin{bmatrix} b_2 & b_3 & \cdots & b_{n_B} \\ -a_1 b_2 + b_3 & & \cdots & -a_1 b_{n_B} \\ \vdots & \vdots & \vdots & \vdots \\ & & \cdots & \end{bmatrix} \begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(t-(n_B-1)) \end{bmatrix} +$$
$$\begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n_A} \\ a_1^2 - a_2 & \cdots & \cdots & a_1 a_{n_A} \\ \vdots & \vdots & \vdots & \vdots \\ & & \cdots & \end{bmatrix} \begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ \vdots \\ y\big(t-(n_A-1)\big) \end{bmatrix} \tag{5}$$

Prediction model is the last P equations of (5), and can be described in compact form

$$y = H_p u_p + H_m u_m + F y_m, \tag{6}$$

where terms of (6) are as appear in (5) respectively. Elements of H_m and F can be described by

$$H_m(i,j) = b_{(i+j)} + \textstyle\sum_{l=1}^{i-1} (-a_l)\, H_m(i-l,j); \quad i = 1{:}P,$$
$j = 1{:}n_B - 1$ and $a_l, b_l = 0$ for $l < 1$, and $l > n_A$ and $l > n_B$, (7-a)

$$F(i,j) = -a_{(i+j-1)} + \textstyle\sum_{l=1}^{i-1} (-a_l)\, F(i-l,j); \quad i = 1{:}P,$$
$j = 1{:}n_A$ and $a_l = 0$ for $l < 1$ and $l > n_A$. (7-b)

For linear systems and unconstrained quadratic cost function (8-a), controller gain matrix K is (8-b) [2]

$$J = Q^T (y-w)^2 Q + \mathcal{R}^T u_p \mathcal{R} \tag{8-a}$$
$$K = \left(H_p^T Q H_p + \mathcal{R}\right)^{-1} H_p^T Q, \tag{8-b}$$

where Q and $\mathcal{R}$ are output error and control effort penalty weighting matrices, and control input to the plant is

$$u(t) = k\big(w - (H_m u_m + F y_m)\big), \tag{9}$$

where $k = [k_1,\ k_2, \cdots,\ k_P]$ is the first row of K in (8), and w is the reference trajectory. When future reference trajectory is known, w is set to: $w = [w(t+d+1), w(t+d+2), \cdots, w(t+d+P)]^T$, and control is named as Preview MPC, otherwise w is set to: $w = [w(t), w(t), \cdots, w(t)]^T$, and control is named as Non-preview MPC.

IV. Basics of Pole-Placement GPC

Using Sections II and III results, we can compare and match counterpart terms in pole-placement and GPC to derive GPC gain. Equation (9) can be re-written as

$$u(t) + kH_m u_m = kw - kFy_m$$

Terms of this equation can be transformed to the z-power series form,

$$u(t) + kH_m u_m = u(t) + kH_m \begin{bmatrix} u(t-1) \\ \vdots \\ u(t-(n_B-1)) \end{bmatrix}$$
$$= [1 \quad kH_m] \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(n_B-1)} \end{bmatrix} u(t), \tag{10-a}$$
$$kw = k \begin{bmatrix} w(t+d+1) \\ \vdots \\ w(t+d+P) \end{bmatrix} = k \begin{bmatrix} z^{d+1} \\ \vdots \\ z^{d+P} \end{bmatrix} w(t), \tag{10-b}$$
$$kFy_m = kF \begin{bmatrix} 1 \\ \vdots \\ z^{-(n_A-1)} \end{bmatrix} y(t). \tag{10-c}$$

In standard pole-placement RST controller, control input is given by

$$S(z^{-1})u(t) = T(z^{-1})w(t) - R(z^{-1})y(t), \tag{11}$$

where R, S, and T were defined in (1).

When the future reference trajectory is known or can be estimated, polynomial T can be in ascending power of z up to prediction horizon P

$$T(z) = t_1 z^{d+1} + \cdots + t_{n_T} z^{d+n_T},\ n_T = P \tag{12}$$

then, terms of (11) and (12) can be represented similar to (10)

$$R(z^{-1})y(t) = [r_0 \quad \cdots \quad r_{n_R}] \begin{bmatrix} 1 \\ \vdots \\ z^{-n_R} \end{bmatrix} y(t), \tag{13-a}$$
$$S(z^{-1})u(t) = [1 \quad \cdots \quad s_{n_S}] \begin{bmatrix} 1 \\ \vdots \\ z^{-n_S} \end{bmatrix} u(t), \tag{13-b}$$
$$T(z)w(t) = [t_1 \quad \cdots \quad t_{n_T}] \begin{bmatrix} z^{d+1} \\ \vdots \\ z^{d+n_T} \end{bmatrix} w(t). \tag{13-c}$$

The corresponding pole-placement and GPC terms are

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$$\text{for } R\text{:}\quad kF\begin{bmatrix}1\\ \vdots\\ z^{-(n_A-1)}\end{bmatrix}=\begin{bmatrix}r_0 \dots r_{n_R}\end{bmatrix}\begin{bmatrix}1\\ \vdots\\ z^{-n_R}\end{bmatrix}, \tag{14-a}$$

$$\text{for S: }\begin{bmatrix}1 & kH_m\end{bmatrix}\begin{bmatrix}1\\ z^{-1}\\ \vdots\\ z^{-(n_B-1)}\end{bmatrix}=\begin{bmatrix}1 \dots s_{n_S}\end{bmatrix}\begin{bmatrix}1\\ z^{-1}\\ \vdots\\ z^{-n_S}\end{bmatrix}, \tag{14-b}$$

$$\text{for } T\text{:}\quad k\begin{bmatrix}z^{d+1}\\ \vdots\\ z^{d+P}\end{bmatrix}=\begin{bmatrix}t_1 \dots t_{n_T}\end{bmatrix}\begin{bmatrix}z^{d+1}\\ \vdots\\ z^{d+n_T}\end{bmatrix}, n_T=P. \tag{14-c}$$

Equations (14-a,b) can be arranged in $Mx=c$ form to calculate GPC gain k.

$$\begin{bmatrix}H^T_{m_{(n_B-1)\times P}}\\ F^T_{n_A\times P}\end{bmatrix}\begin{bmatrix}k_1\\ \vdots\\ k_P\end{bmatrix}=\begin{bmatrix}s_1\\ \vdots\\ s_{n_S}\\ --\\ r_0\\ \vdots\\ r_{n_R}\end{bmatrix}\quad\begin{matrix}n_S=n_B-1\\ n_R=n_A-1\\ n_T=P\\ t_j=k_{i,j}, j=1:P\\ i=1:P\end{matrix} \tag{15}$$

For the desired RST controller to have a GPC counterpart, (15) should have at least one solution. There exists a solution for k when $[H_m|F]^T$ is full row rank; i.e., row rank of $[H_m|F]^T$ be equal to $n_A+(n_B-1)$, and $P\geq n_A+(n_B-1)$. But as will be shown in Theorem 1, the row rank of $[H_m|F]^T$ is equal to $\max(n_A,n_B-1)$. Therefore, an RST controllers has a GPC counterpart when the order of desired closed loop characteristic polynomial be $n_{\bar{A}}\leq\max(n_A,\ n_B-1)$ and is not the same as (4) for general pole placement.

Theorem 1: For co-prime A and B polynomials, the row rank of augmented matrix $[H_m|F]^T$ is $\max(n_A,n_B-1)$.

Proof: Equations (7) explicitly shows that the first terms $b_{(i+j)}=0$ for $i>n_B-1$, and $a_{(i+j-1)}=0$ for $i>n_A$ respectively, and following rows are linear combination of previous rows. Thus we can conclude that

$$\text{column rank of}\begin{bmatrix}H^T_{m\,(n_B-1)\times P}\\ ---\\ F^T{}_{n_A\times P}\end{bmatrix}=\max(n_A,n_B-1) \tag{16}$$
$$for\ P\geq\max(n_A,n_B-1)$$

The result of Theorem 1 puts some restrictions on R and S. However, requirements such as integral control can easily be considered in predefined parts of $R(z^{-1})$ and $S(z^{-1})$. ■

Theorem 2: Every pole-placement controller has a corresponding GPC counterpart when its closed-loop characteristic equation $\bar{A}(z^{-1})$ be

$$n_{\bar{A}}\leq\max(n_A,n_B-1) \tag{17}$$

Proof: Replacing unknowns vector of (2) with (15)

$$\begin{bmatrix}1 & \vdots & 0 & b_1 & \vdots & 0\\ a_1 & \vdots & 0 & b_2 & \vdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ a_{n_A-1} & \vdots & 0 & b_{n_B-1} & \vdots & 0\\ a_{n_A} & \vdots & 1 & b_{n_B} & \vdots & b_1\\ 0 & \vdots & a_1 & 0 & \vdots & b_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \vdots & a_{n_A} & 0 & \vdots & b_{n_B}\end{bmatrix}\begin{bmatrix}H^T_{m_{(n_B-1)\times P}}\\ F^T_{n_A\times P}\end{bmatrix}\begin{bmatrix}k_1\\ \vdots\\ k_P\end{bmatrix}=\begin{bmatrix}\bar{a}_1-a_1\\ \bar{a}_2-a_2\\ \vdots\\ \bar{a}_{n_A}-a_{n_A}\\ \bar{a}_{n_A+1}\\ \vdots\\ \bar{a}_{n_{\bar{A}}}\end{bmatrix} \tag{18}$$

The first term of (18) is a full rank square n_A+n_B-1 matrix, and second one is $(n_A+n_B-1)\times P$ rectangular matrix, and its column rank is $\max(n_A,n_B-1)$. Therefore their product will be an $(n_A+n_B-1)\times P$ rectangular matrix, and its rank is equal to $\max(n_A,n_B-1)$. This means that degrees of freedom of desired closed loop characteristic polynomial $\bar{A}(z^{-1})$ is $\max(n_A,n_B-1)$.

However, is this restriction on the pole places, or number of poles? Surprisingly, as will be shown, the product of the first two matrices of (18) is a simple matrix, and its rows after $\max(n_A,n_B-1)$ are identically zero, i.e., order of $\bar{A}(z^{-1})$ is $\max(n_A,n_B-1)$. Two first terms in (18) can be expanded as:

$$M\begin{bmatrix}H^T_m\\ -\\ F^T\end{bmatrix}=\left[M1_{(n_A+n_B-1)\times(n_B-1)}\middle|M2_{(n_A+n_B-1)\times(n_A)}\right]$$

$$\begin{bmatrix}H^T_{m_{(n_B-1)\times P}}\\ F^T_{n_A\times P}\end{bmatrix}=M1_{(n_A+n_B-1)\times(n_B-1)}H^T_{m_{(n_B-1)\times P}}$$
$$+M2_{(n_A+n_B-1)\times(n_A)}F^T_{n_A\times P}$$

Replacing terms of final expanded form by (3) and (5) and focusing on last $\big((n_A+n_B-1)-\max(n_A,n_B-1)\big)$ rows, named X, gives

$$X=$$
$$\begin{bmatrix}a_{\max(n_A,n_B-1)} & & \\ 0 & \ddots & a_{n_A-1}\\ 0 & \dots & a_{n_A}\end{bmatrix}\begin{bmatrix}b_2 & -a_1b_2+b_3 & \dots\\ \vdots & \vdots & \\ b_{n_B-1} & -a_1b_{n_B-1}+b_{n_B} & \vdots\\ b_{n_B} & -a_1b_{n_B} & \dots\end{bmatrix}$$
$$+\begin{bmatrix}b_{\max(n_A,n_B-1)} & & \\ 0 & \ddots & b_{n_B-1}\\ 0 & \dots & b_{n_B}\end{bmatrix}\begin{bmatrix}-a_1 & a_1^2-a_2 & \dots\\ \vdots & \vdots & \vdots\\ -a_{n_A-1} & \dots & \vdots\\ -a_{n_A} & a_1a_{n_A} & \dots\end{bmatrix},$$

$a_i=0, i>n_A,$ and $b_j=0, i>n_B$

Reduced $M1$ and $M2$ matrices are upper triangular, and calculation shows that the first column of X equals zero

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$$X(:,1) = \begin{bmatrix} \vdots \\ a_{n_A}b_{n_B-1} + a_{n_A-1}b_{n_B} - b_{n_B}a_{n_A-1} - b_{n_B-1}a_{n_A} \\ a_{n_A}b_{n_B} - b_{n_B}a_{n_A} \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 0 \end{bmatrix}$$

Equation (7-a, b) shows that the proceeding columns of H_m^T and F^T are the same linear combinations of the previous columns plus the first column shifted upward. Thus, all rows after max $(n_A, n_B - 1)$ of multiplier matrix in (18) are equal to zero. ∎

A. *Offset free control*

When the plant is Type 1, or a differentiator is added as a predefined part of $S(z^{-1})$, we expect an offset free control, i.e., the DC-gain of $R(z^{-1})$ and $T(z)$ should be equal;

$$\sum_{i=1}^{P} k_i = R(z^{-1})|_{z=1}, or \ \sum_{i=1}^{P} k_i = \sum_{i=1}^{n_R} r_i. \quad (19)$$

Theorem 3: When there is an integrator in forward route, DC gain of $R(z^{-1})$ and $T(z)$ ($\sum_{i=1}^{P} k_i$) are equal, and the derived control is offset free.

Proof: When the plant transfer function is Type 1 or greater, the steady state gain of the system is infinity; i.e. $A(z^{-1})_{|z=1} = 1 + a_1 + \cdots + a_{n_A} = 0$, or

$$a_1 + \cdots + a_{n_A} = -1. \quad (20)$$

Relation between elements of polynomial R and k can be highlighted from upper set equations in (15);

$$(F^T)_{n_A \times P} \begin{bmatrix} k_1 \\ \vdots \\ k_P \end{bmatrix} = \begin{bmatrix} r_0 \\ \vdots \\ r_{n_R} \end{bmatrix}, n_R = n_A - 1,$$

and (19) is true when sum of elements of each column of F^T be equal to 1. Replacing for $F^T(i,j)$'s from (7-b) and renumbering row index for corresponding $F(i,j)$'s, one obtains

$$\sum_{j=1}^{n_A} F^T(j,i) = \sum_{j=1}^{n_A} F(i,j) = \sum_{j=1}^{n_A} \left(-a_{(i+j-1)} - \sum_{l=1}^{i-1} a_{(i-l)} F(i-l,j)\right).$$

For $i = 1$, from (7-b)

$$\sum_{j=1}^{n_A} F^T(j,1) = \sum_{j=1}^{n_A} F(1,j) = \sum_{j=1}^{n_A} -a_j = 1,$$

and for $i \geq 2$, for succeeding rows we can conclude consecutively using (20)

$$\sum_{j=1}^{n_A} F^T(j,i) = \sum_{j=1}^{n_A} F(i,j) = \left(-a_i - \cdots - a_{n_A}\right) + \sum_{l=1}^{i-1} -a_{(i-l)} \sum_{j=1}^{n_A} F(i-l,j) = \left(-a_i - \cdots - a_{n_A}\right) + (-a_1 - \cdots - a_{i-1}) = 1.$$ ∎

V. DESIGN of UNCONSTRAINED POLE-PLACEMENT GPC

In conventional unconstrained GPC design, gain matrix (8), is used to calculate the controller gain. In Section IV we derived some equality constraint on gain vector k to satisfy the desired closed-loop characteristic. In this section we show that GPC optimization problem can be modified to be directly solved for K matrix with some equality constraints on its first row. Cost function (8-a) for basic GPC can be written as

$$J = \left(w - (H_p u_p + f)\right)^T Q \left(w - (H_p u_p + f)\right) + u_p^T \mathcal{R} u_p,$$
$$f = H_m u_m + F y_m. \quad (21)$$

Unconstrained optimization on input vector u_p leads to a gain matrix K. There are two approaches to consider constraints (15); the first is to consider it for the first row of the GPC gain matrix (8), and second approach is to consider it for all rows of K. In practice the first row determines the GPC gain so we consider constraints on first row. Control at time t is given by (9), or $u(t) = (w-f)^T [k_1 \ k_2 \ \cdots \ k_P]^T$. Thus u_p can be written as

$$u_p = \begin{bmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+P-1) \end{bmatrix} = \begin{bmatrix} (w-f)^T & 0_{1\times(P-1)} \\ 0_{(P-1)\times P} & I_{(P-1)} \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_P \\ u(t+1) \\ \vdots \\ u(t+P-1) \end{bmatrix} =$$
$$WF_{P\times(2P-1)} \bar{u}_{P\,(2P-1)\times 1}. \quad (22)$$

Rewriting cost function (21) in standard quadratic form

$$J = u_p^T \left(H_p^T Q H_p + \mathcal{R}\right) u_p - u_p^T H_p^T Q (w-f) - (w-f)^T Q H_P u_p + (w-f)^T Q (w-f)$$

and replacing for u_p from (22) gives the cost function J as a standard quadratic function of $\bar{u}_p$

$$J = \bar{u}_p^T W F^T \left(H_p^T Q H_p + \mathcal{R}\right) W F \bar{u}_p - \bar{u}_p^T W F^T H_p^T Q^T (w-f) - (w-f)^T Q H_p W F \bar{u}_p + (w-f)^T Q (w-f). \quad (23)$$

When (23) be solved subject to constraints (22) on k, the derived GPC controller will have desired closed-loop poles. At every time step t, optimal command $u(t)$ is calculated indirectly using calculated k part of $\bar{u}_p$.

This algorithm also can be used for designing GPC without pre-specified closed loop characteristics when solved without equality constraints (22). For positive definite Q and $\mathcal{R}$, (23) is a convex optimization problem. Although the WF in (22) is varying with time but because constraints (22) are satisfied through optimization, close loop characteristic of the controller is time invariant. Effect of time variance of the controller gain and other important features of the proposed algorithm and the previous sections results are discussed below.

A. *Discussion 1*

1. A pole placement designed RST controller has a GPC counterpart if the order of its closed loop characteristic polynomial meet (17), i.e., $n_{\bar{A}} \leq \max(n_A, n_B - 1)$.
2. Regarding proposed pole-placement GPC design algorithm, the number of optimization variable (k_i's) put in place of $u(t)$, are equal to P. Pole placement requirements puts max $(n_A, n_B - 1)$ equality constrains on optimization variables and there remain $P - \max(n_A, n_B -$

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1) degrees of freedom to be used for time optimality, but it is only for preview control.

3. According to (15), counterpart $R(z^{-1})$ and $S(z^{-1})$ polynomials are fixed and do not change by changing MPC design parameters P, Q, and $\mathcal{R}$.

4. Control horizon (M , $M \leq P$) also does not change closed loop characteristic. Changing M do not changes anything for non-preview control, but it changes tracking characteristic of the control for preview control when $P > \max{(n_A, n_B - 1)}$.

5. Preview vs. Non-preview pole-placement GPC: As clarified in previous discussions, GPC design parameters can only alter $T(\mathrm{z})$. But according to (19), $T(\mathrm{z})$ is also fixed for non-preview control. But in preview control, GPC design parameters forms $T(z)$, so tracking behavior of the control. This enables multi-objective optimization of control for both time domain and frequency domain requirements.

6. Is the outcome controller linear or non-linear? Also, is it time variant or invariant? Equations (9)-(14) explicitly show that the outcome controller is equivalent to the form of (11), where $R(z^{-1})$ and $S(z^{-1})$ are set to desired pole-placement controller terms. Nevertheless, $T(z)$ may vary with time for pre-view controller and for constrained control, which will be discussed in Section VI. However, GPC is a finite time optimal control, and is time-variant as any other finite time optimal control. However, $T(z)$ is the time varying term and do not alter closed loop characteristic.

7. Stability of the control: For unconstrained pole placement GPC, (23) always has at least one finite solution. For preview control, although the controller may be time variant, but control is bounded input-bounded output stable. This can be proved easily by expanding the control as sum of bounded terms of $T(z)$ (i.e., $k_i z^i$) of the stable closed loop system

$y(t) = \frac{T(z)B(z)}{A(z)S(z)+B(z)R(z)} r(t)$, $T(z) = k_1 z^1 + \cdots + k_p z^p$,

so existence of finite $k_i' s$, as there are for consistent constraints, is sufficient for stability of the closed loop control system.

8. Comparison with others' works

Some other researchers, e.g., [18] have used (8-b) instead of (22) which leads to complicated results to solve for GPC error and control effort weighting. Also they have not clearly discussed important issues such as degrees of freedom and limitations on desired closed loop characteristics, and time optimality consideration along with desired closed loop features.

Works of Maciejowski [19], Cairano and Bemporad in [1] followed by Hartley and Maciejowski in [11] are based on development of state observer-based of (23) to find all MPC design parameters. But they also have not addressed the above-mentioned important issues. Our solution in frequency domain, based on transfer function description, is also more amenable for studying GPC related problems.

VI. Design of Constrained Pole-Placement GPC

The most favorable feature of model predictive controllers is their ability to handle constraints on input, states and output. GPC formulation derived above can also consider these constraints. Constraints on input u_p can be transformed to constraints on new defined variable $\bar{u}_p$,

$$\begin{bmatrix} u_{p\,\min}(t) \\ u_{p\,\min}(t+1) \\ \vdots \\ u_{p\,\min}(t+P) \end{bmatrix} \leq \begin{bmatrix} u_p(t) \\ u_p(t+1) \\ \vdots \\ u_p(t+P) \end{bmatrix} \leq \begin{bmatrix} u_{p\,\max}(t) \\ u_{p\,\max}(t+1) \\ \vdots \\ u_{p\,\max}(t+P) \end{bmatrix}. \quad \text{(24-a)}$$

Replacing u_p using (22), results in constraints on $\bar{u}_p$

$$\begin{bmatrix} u_{p\,\min}(t) \\ u_{p\,\min}(t+1) \\ \vdots \\ u_{p\,\min}(t+P) \end{bmatrix} \leq \begin{bmatrix} (w-f)^T & 0_{1\times(P-1)} \\ 0_{(P-1)\times P} & I_{(P-1)} \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_P \\ u(t+1) \\ \vdots \\ u(t+P-1) \end{bmatrix} \leq \begin{bmatrix} u_{p\,\max}(t) \\ u_{p\,\max}(t+1) \\ \vdots \\ u_{p\,\max}(t+P) \end{bmatrix}. \quad \text{(24-b)}$$

Based on (24-b), control input constraints can be transferred to inequality constraint in the form of $Ax < b$. Constrained GPC controller with pre-specified closed-loop characteristic can be derived by minimizing cost function (23) subject to pre-specified closed-loop requirements (21) and input constraints specified by (24-b).

Constraints on output y, can also be transformed to constraints on input u_p, and afterward on new defined $\bar{u}_p$ using prediction model (9).

For equality and inequality constrains $y = y_0$ *and* $\underline{y} \leq y \leq \bar{y}$, we have

$$H_p u_p + f = y_0 \Rightarrow H_p u_p = y_0 - f, \quad \text{(24-c)}$$

$$\underline{y} \leq H_p u_p + f \leq \bar{y} \Rightarrow \begin{bmatrix} H_p \\ -H_p \end{bmatrix} u_p \leq \begin{bmatrix} \bar{y} - f \\ -\underline{y} + f \end{bmatrix}, \quad \text{(24-d)}$$

which can be considered in (24-a) and (24-b) as well. However, existence of a feasible solution set to the optimization problem should be checked.

B. Stability of the constrained pole-placement GPC

Stability of the constrained pole-placement GPC also can be shown similar to unconstrained control. When constraints are consistent, optimization problem (23) has bounded solution, and control is BIBO stable.

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VII. REFERENCE GOVERNOR GPC

Reference governor (RG) is an add-on control schemes to stable control loops, which will be activated when changing reference input is going to deteriorate constraints. With the proposed designing algorithm, one can freeze $S(z^{-1})$ and $R(z^{-1})$ of a linear unconstraint GPC - a pole-placement GPC - and let $T(z)$ to act as a dynamic RG. Thus RG functionality is an intrinsic capability of pole-placement GPC.

VIII. ILLUSTRATIVE EXAMPLES

Findings of the paper are illustrated in some illustrative examples.

Example 1: Order of closed loop characteristic polynomial

Example 1 shows basic results of sections II-IV. Consider below plant transfer function, for which $n_A = 2$ and $n_B = 2$,

$G(z^{-1}) = \frac{y(z^{-1})}{u(z^{-1})} = \frac{0.2z^{-1}+0.3z^{-2}}{1-1.8z^{-1}+0.8z^{-2}}$.

To design an RST controller using (2) and find a unique minimal solution, one should consider (4)

$n_R = n_A - 1 = 1 \quad \Rightarrow \quad R = r_0 + r_1 z^{-1}$,

$n_S = n_B - 1 = 1 \quad \Rightarrow \quad S = 1 + s_1 z^{-1}$,

$n_{\bar{A}} \le n_A + n_B - 1 = 3$,

and to have the counterpart GPC controller one should consider (17)

$n_{\bar{A}} \le \max(n_A, n_B - 1) = 2$.

We select a second order characteristic equation as follows

$\bar{A}(z^{-1}) = 1 - 1.2z^{-1} + 0.52z^{-2}$.

Polynomials R and S can be determined using (2)

$$\begin{bmatrix} 1 & 0.2 & 0 \\ -1.8 & 0.3 & 0.2 \\ 0.8 & 0 & 0.3 \end{bmatrix} \begin{bmatrix} s_1 \\ r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} -1.2+1.8 \\ 0.52-0.8 \\ 0 \end{bmatrix}.$$

The resulting polynomials are

$S = 1 + 0.3078z^{-1}, R = 1.4609 - 0.8209z^{-1}$.

GPC prediction horizon should be selected using (16; i.e., $P \ge 2$. For $P = 4$, (15) will be

$$\begin{bmatrix} 0.3 & 0.54 & 0.732 & 0.8856 \\ 1.8 & 2.44 & 2.952 & 3.3616 \\ -0.8 & -1.44 & -1.952 & -2.3616 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0.3078 \\ 1.4609 \\ -0.8209 \end{bmatrix} \tag{25}$$

Ranks of both coefficients and its augmented matrices in (25) are equal to 2 and satisfy (16). For $P = 2$ it has unique solution: $[k_1 \; k_2] = [0.1577 \; 0.4824]$. Plant is Type 1 and for offset free control (19), $R(z^{-1})_{z=1} = k_1 + k_2 = 0.64$ should be satisfied. For $P > 2$, we have $P - 2$ degrees of freedom for k_i's, which can be a solution of the optimization problem in the corresponding GPC subject to equality constraints (18).

If one consider (4) instead of (17) to design RST controller, the resulting controller has not a counterpart GPC. For example, for $n_{\bar{A}} \le n_A + (n_B - 1) = 3$ and $\bar{A}(z^{-1}) = (1 - 1.2z^{-1} + 0.52z^{-2})(1 - 0.9z^{-1})$, polynomials R and S can be found using (3) as

$$\begin{bmatrix} 1 & 0.2 & 0 \\ -1.8 & 0.3 & 0.2 \\ 0.8 & 0 & 0.3 \end{bmatrix} \begin{bmatrix} s_1 \\ r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} -2.1+1.8 \\ 1.6-0.8 \\ -0.468 \end{bmatrix}.$$

Solving this equation results in

$R = 0.5374 - 0.4734z^{-1}, S = 1 - 0.4075z^{-1}$.

For $P = 4$, according to (33), k should be a solution of

$$\begin{bmatrix} -0.8 & -1.44 & -1.952 & -2.3616 \\ 1.8 & 2.44 & 2.952 & 3.3616 \\ 0.3 & 0.54 & 0.732 & 0.8856 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} -0.4734 \\ 0.5374 \\ -0.4075 \end{bmatrix}.$$

Since ranks of coefficients matrix is 2, and rank of its augmented matrix is 3, then there is not any solution for k.

Example 2: Minimum prediction horizon

Example 2 illustrates design of pole-placement GPC for minimum prediction horizon. When controller is designed subject to the desired closed-loop poles considered in (20), the controller gain matrix can be calculated using (32)-(33). Step response, optimum k_i's, and closed loop characteristic polynomial coefficients for desired characteristic polynomial $\bar{A}(z^{-1}) = 1 - 1.2z^{-1} + 0.52z^{-2}$, $P = M = \max(n_A, n_B - 1) = 2$, $Q = \mathcal{R} = I$ are shown in Fig. 2 for unconstrained preview, and in Fig.3 for non-preview control. As can be seen there is no extra degree of freedom for k_i's, and results for both are the same, and $\sum k_i' s = 0.64$, the same as in Example 1.

Example 3: Effect of greater prediction horizon

Figure 4 shows output results for prediction horizon $P = 4 > P_{\min}$, and for $Q = I$ and $\mathscr{R} = I$. Although they vary with time due to remaining two degrees of freedom for k_is, but closed loop characteristic is as desired.

Example 4: Effect of GPC design parameters on preview control

Figure 5 shows simulation results of Example 3 by decreasing control effort weighting to $\mathcal{R} = 0.01I$. As we expect, it has faster tracking behavior with bigger control, while reserving regulating behavior (closed loop characteristic unchanged). Both Figures show that although k_i-s may vary during steady state, but controls do not change, and variations are due to degrees of freedom of k_i-s.

Example 5: Constrained pole-placement GPC

Figures 6 shows input constrained control of preview control of the plant for $\mathscr{R} = 0.01I$, shown in Fig. 4, which has meet control constraint while preserving closed loop characteristic. But for Non-preview there is no feasible solution when constraint is activated. Figure 7 shows outcomes of the algorithm for a 200 times iteration, which is not converged, and unacceptable solution.

IX. CONCLUSION

In this paper we introduced a new formulation for GPC to be comparable to RST controller. By comparing two formulation peer to peer, we derived a new formulation for designing GPC with pre-specified closed-loop properties. The derivations explicitly show some important properties of the GPC; some has not been clarified before. The developed method can also be used for input and output constrained control with guaranteed close loop stability; which has not addressed before. New derived properties of GPC and capabilities of the proposed design method are examined through some examples. The approach is also attractive for theoretical studies; it unifies some other extension and reduces them to basic MPC context such as controller matching which is not a new problem, and can be solved as an ordinary MPC. We showed also that a GPC can intrinsically be a reference governor controller, with guaranteed closed loop stability; a seamless solution to the problem rather than add-on schemes.

X. FUTURE WORKS

As in similar works, to have more flexibility, and optimize closed loop characteristics along with time response behavior, loosing equality constraints will be considered in future works. Extending the proposed method to MIMO systems is also an interesting subject.

Our main motivation to this development has been preparing required analysis methods and design tools for model reference MPC for data driven MPC, an emerging field in control engineering. So, we will demonstrate some new application of outcomes in our ongoing works.

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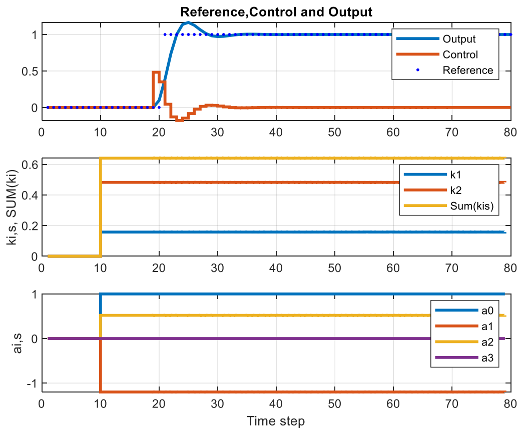

Figure 2: Pole-placement GPC, unconstrained, preview control, P=2, R=1.

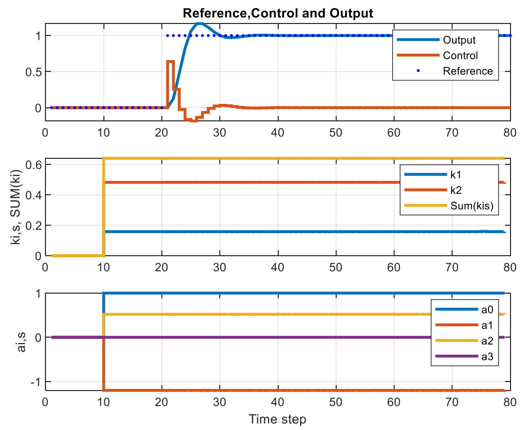

Figure 3: Pole-placement GPC, unconstrained non-preview control, $\mathrm{P} = 2$, $\mathcal{R} = 1$.

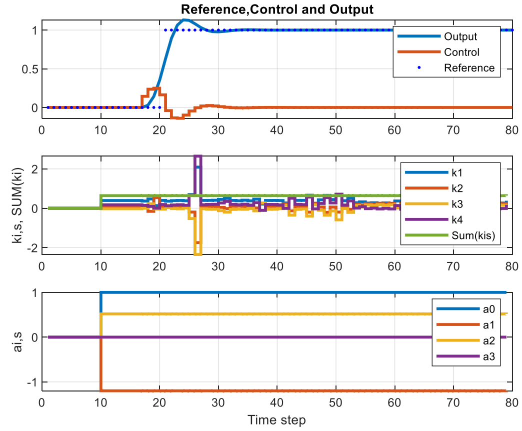

Figure 4: Pole-placement GPC, unconstrained preview control, P=4, R=1

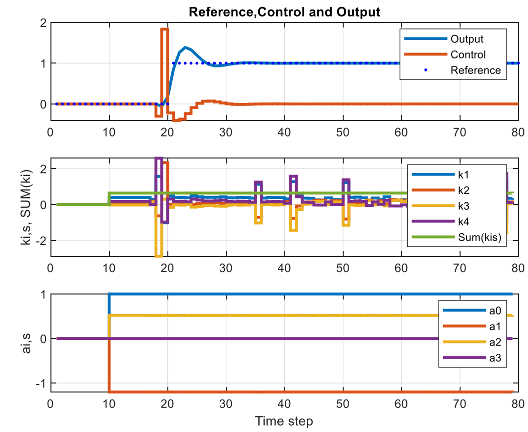

Figure 5: Pole-placement GPC, unconstrained preview control, P=4, R=0.01.

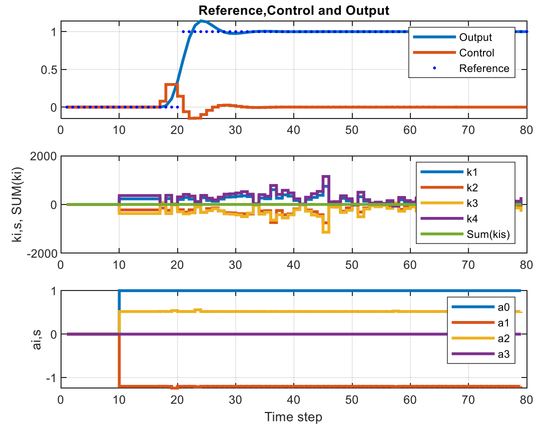

Figure 6: Pole-placement GPC, constrained preview control, $\mathrm{P} = 4$, $\mathcal{R} = 0.01$.

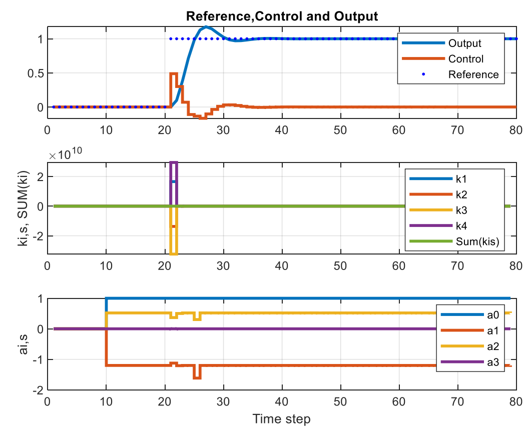

Figure 7: Pole-placement GPC, constrained, non-preview control, P=4, R=0.01. (Not converged)

Note: In all figures, the upper one is for Reference, Control and Output, the middle shows calculated controller gain vector and, and sum of the gain elements, and the bottom shows closed loop characteristic equation's coefficients.

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