Design of adaptive backstepping control for stabilization and synchronization of a class of uncertain fractional-order chaotic systems with uncertainties and disturbances

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Abstract— **In this work, the goal is to achieve both stabilization and synchronization of a general class of fractional-order chaotic systems. It is assumed that there are uncertainties and external disturbances in the system, and it is also supposed that the system parameters are unknown. Uncertainties and disturbances are undesirable factors that can disrupt the system response. To this end, appropriate adaptive laws have been proposed to address these factors. A systematic step-by-step technique is also developed for designing a controller based on the backstepping method. The analysis of the proposed control structure is carried out according to the fractional Lyapunov theorem which is a more realistic technique for the analysis and stability of nonlinear systems. Finally, the simulation results are presented to confirm and prove the effectiveness of the proposed method. The results of implementing our proposed controller for different fractional-order chaotic systems are compared with some control approaches in the available papers and it confirms the superiority of the proposed control in this paper.**

***Keywords*: Adaptive control, backstepping control, chaotic system, fractional calculus, synchronization.**

I. INTRODUCTION

IN recent decades, much attention has been paid to nonlinear control. Given the chaos in real-world systems and many useful applications in the fields of physics and engineering, several methods have been proposed in the last few years to stabilize and synchronize chaotic systems such as: fuzzy control [1,2], observer control [3], sliding mode control [4-6], active control [7], output feedback control [8,9], impulsive control [10,11], neural network control [12,13], etc.

Fractional-order chaotic systems have also been developed. Although fractional calculus is an old mathematical subject dating back more than 300 years, it has attracted the attention of many researchers in recent years. In fact, all physical phenomena in nature exist in the form of fractional-order and the integer-order differential equation is just a particular case of fractional-order differential equation. Today, many fractional-order differential systems have chaotic behavior. The advantages of fractional-order models in comparison to integer-order models are: first, the fractional description can provide a more explicit and accurate explanation, so it is closer to reality. Second, memory is included in fractional-order systems. Third, fractional-order models can enlarge the key space, and hence are more efficient in coding because they have more customizable variables. The various techniques mentioned above have also been used to stabilize and synchronize fractional-order chaotic systems [14-19].

In [20], Huang et al. have developed an active control method for the synchronization and anti-synchronization of the fractional-order chaotic financial systems. In [21], a sliding mode control method, which is limited to three-dimensional system, is presented to synchronize the fractional-order chaotic system. Muthukumar et al. have proposed the fuzzy predictive control for synchronizing two similar systems in the T–S fuzzy model [22]. Chen et al. have used neural networks to synchronize the

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fractional-order system [23]. Most researches regarding the stabilization and synchronization of fractional-order chaotic systems have not taken the uncertainties and disturbances into account and have also assumed all system parameters to be known. However, the proposed method is applicable to both synchronization and stabilization of chaotic systems with uncertainties, disturbances, and unknown parameters.

In many real systems, the variability of time, the uncertainty of dynamics and the presence of noise cause uncertainties and disturbances in the systems and it is difficult to determine uncertain bounds and disturbances in advance. Furthermore, all system parameters may not be specified. Therefore, adaptive control [24-27], which is another idea of this article, is suggested to overcome these problems.

In [28], the problems of synchronization and stabilization of fractional-order chaotic systems in the presence of a fractional sliding mode controller have been investigated by Aghababa. The control signal in [28] has large fluctuations and this is not practical. In [29] and [30], chattering occurs in the control signal, which restricts the operation of the controller. However, another strength of this paper is that permanent chattering is removed in the control signal and it is practical.

One of the most popular methods used to stabilize and synchronize nonlinear systems is the backstepping method developed by kristic et al [31]. Backstepping control is a recursive method that combines Lyapunov function with feedback control design. This method converts the whole system design problem into several successive design problems for lower-degree subsystems and even scalars. Due to the flexibility of subsystems with lower degree, a One of the most popular methods used to stabilize and synchronize nonlinear systems is the backstepping method developed by kristic et al [31]. Backstepping control is a recursive method that combines Lyapunov function with feedback control design. This method converts the whole system design problem into several successive design problems for lower-degree subsystems and even scalars. Due to the flexibility of subsystems with lower degree, a backstepping control has the capability of solving stabilization, synchronization and robust control problems under freer constraints than other methods. A control algorithm is chosen for each step in a way that the corresponding Lyapunov function expresses the stability of each system. The extension of Lyapunov's theory to fractional-order nonlinear systems along with the development of Mittag-Leffler concept of stability is proposed by Li et al [32].

Using this stability concept in order to design controller for fractional-order nonlinear systems is an interesting topic which is the focus of this paper. Although the stability of systems in [29] and [33,34] has been proven by traditional Lyapunov theorem, it is not applicable to fractional-order systems. However, in this analysis, the stability analysis and therefore the design of the stabilization and synchronization controller are combined with the direct fractional Lyapunov method and Mittag-Leffler stability which offer a more realistic approach to stability evaluation of systems.

The proposed design in [34] is based on an adaptive active sliding mode controller for the synchronization of two integer-order and fractional-order chaotic systems that are limited to synchronizing two-dimensional systems. In [35], Nikdel et al. have developed an adaptive backstepping control scheme to stabilize two-dimensional chaotic systems. However, the control design presented in this article is applicable to n-dimensional systems. The proposed method in [36] uses the backstepping method to synchronize two similar fractional-order chaotic systems and also considers all the system parameters as known parameters. However, in this paper, though, the proposed backstepping control scheme can be used to synchronize two similar and two different systems.

Using the combination of fractional Lyapunov stability and Mittag-Leffler stability for backstepping-based control of fractional-order chaotic systems has been understudied. Moreover, most of the proposed controllers use the traditional Lyapunov stability theory. This fact has provided the motivation to design controllers for fractional-order systems using the fractional Lyapunov stability method. According to the above discussion, in this paper, an adaptive backstepping control method is used for each n-dimensional fractional-order system. The proposed approach addresses both problems of stabilization and synchronization of fractional-order chaotic systems. First, the stabilization of fractional-order chaotic systems with unknown parameters, uncertainties and disturbances is studied showing that the states of system tend to zero. Then, utilizing the suggested controller, the synchronization of fractional-order chaotic systems is examined. Furthermore, appropriate adaptive laws have been developed to deal with unknown parameters. Finally, using fractional Lyapunov theory, the convergence and stability of the proposed method are investigated.

The advantages of our proposed method are as follows:

- The proposed method is applicable to a wide range of chaotic systems.
- chattering phenomenon is entirely eliminated in the proposed method and it is practically usable.
- The proposed method is applicable to both synchronization and stabilization of n-dimensional fractional-order chaotic systems

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in the presence of uncertainties, disturbances, and unknown parameters.

- Appropriate adaptive laws are introduced for dealing with unknown parameters. Additionally, information about disturbance bound is not needed in the above method.
- It can be used in applications such as complex networks, secret signaling, multilateral communications and many other engineering fields.

This paper is organized as follows. In section 2, some preliminaries of fractional calculus are briefly reviewed. Section 3 introduces the stabilization issue of fractional-order chaotic systems using adaptive backstepping control. Then, according to Lyapunov theorem, the stability of the proposed approach is investigated. Section 4 explains the synchronization of fractional-order chaotic systems via backstepping approach in the presence of uncertainties, disturbances and unknown parameters. In section 5, numerical simulations show that the suggested techniques are effective and applicable. In this section, three recently published control approaches are also simulated and their results are compared with this proposed method to demonstrate the effectiveness of the proposed controller. The concluding part is in section 6.

II. Preliminaries Of Fractional Calculus

In this paper, the fractional-order derivative/integral operation is shown by the operator ${}_aD_t^q$, expressed as

$$ {}_aD_t^q = \begin{cases} \frac{d^q}{dt^q} & q > 0 \\ 1 & q = 0 \\ \int_a^t (d\tau)^{-q} & q < 0 \end{cases}, \tag{1} $$

where a, t are the limits of operation and $q \in (0,1)$ is the fractional-order [37]. There are many definitions of fractional derivative among which the definitions presented by Caputo, Riemann–Liouville (RL) and Grunwald–Letnikov (GL) are well-known definitions of fractional derivative.

Definition 1. The Riemann–Liouville fractional integration of order q of a continuous function $f(t)$ is represented below

$$ {}_{t_0}I_t^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{1-q}} d\tau, \tag{2} $$

where $\Gamma(q)$ is the well–known Gamma function [16,37].

Definition 2. The q-order Riemann–Liouville fractional derivative of function $f(t)$ is expressed by

$$ {}_{t_0}D_t^q f(t) = \frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{q-m+1}} d\tau, \tag{3} $$

where $m-1 < q \le m$, $m \in N$ [16,37].

Definition 3. The q-order Caputo fractional derivative of function $f(t)$ is given by

$$ {}_{t_0}D_t^q f(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_{t_0}^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q-m+1}} d\tau, & m-1 < q < m \\ \frac{d^m f(t)}{dt^m}, & q = m \end{cases}, \tag{4} $$

where m is the smallest integer number, larger than q [16,37].

Remark 1. In this article, the Caputo's definition of fractional derivative is utilized. For simplicity, the symbol $\boldsymbol{D^q}$ represents fractional derivative.

Theorem 1. Suppose that $x(t) = 0$ is an equilibrium point of the fractional-order nonlinear system

$$ {}_0^C D_t^q x(t) = f\big(t, x(t)\big). \tag{5} $$

If there exists a Lyapunov function $V(t, x(t))$ and class K functions g_i, $i = 1,2,3$ such that

$$ g_1(\|x(t)\|) \le V\big(t, x(t)\big) \le g_2(\|x(t)\|), \tag{6} $$
$$ {}_0^C D_t^q V\big(t, x(t)\big) \le -g_3(\|x(t)\|), \tag{7} $$

then (5) will be asymptotically stable [38].

Lemma 1. Let $x(t) \in R$ be a real-valued continuous differentiable function. Then for any $\mu = 2^n$, $n \in N$,

$$ D_t^q x^\mu(t) \le \mu x^{\mu-1}(t) D_t^q x(t), \tag{8} $$

where $q \in (0,1)$ is the fractional-order [39].

Corollary 1. Let $x(t) \in R$ be a real-valued continuous differentiable function. Then for any time t [39]

$$ \frac{1}{2} D_t^q x^2(t) \le x(t) D_t^q x(t) \ \ \forall q \in (0,1). \tag{9} $$

III. Stabilization Of Fractional-Order Chaotic Systems Using Adaptive Backstepping Control

A. Problem Formulation

Consider a fractional-order chaotic system expressed by the following class of uncertain n-dimensional nonlinear equations

$$ \begin{cases} D^q x_i = x_{i+1}, & 1 \le i \le n-1 \\ D^q x_n = f(x,t) + \boldsymbol{\delta}^T F(x,t) + \Delta f(x,t) + d(x,t) + u(t) \end{cases}, \tag{10} $$

where $q \in (0,1)$, $x(t) = [x_1, x_2, \dots, x_n]^T \in R^n$ is the state vector. $f(x,t) \in R$ and $F(x,t) \in R^{1\times p}$ are known nonlinear functions. $\boldsymbol{\delta} \in R^p$ is the uncertain parameter vector. $\Delta f(x,t) \in R$ and $d(x,t) \in R$ indicate uncertainty and external disturbance, respectively. $u(t) \in R$ is the control input to be designed later.

Assumption 1. The uncertainty and disturbance are bounded and defined by

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$$|\Delta f(x,t)| \leq \sigma \text{ and } |d(x,t)| \leq \vartheta \,. \tag{11}$$

Assumption 2. The constants σ and ϑ are unknown positive.

Definition 4. The purpose of the stabilization problem is to select an appropriate controller $u(t)$ which $\lim_{t\to\infty}\|x(t)\| = 0$, i.e. the states of system (10) will tend to zero [40].

A control algorithm will be developed in the next section.

B. Controller Design

Here, adaptive backstepping control approach is used to get stabilization of a fractional-order chaotic system in (10) with uncertainties, disturbances and unknown parameters.

In each step of the proposed algorithm, a virtual controller is developed for each subsystem and moving to the last equation, the final controller $u(t)$ is optained. In each step, a suitable Lyapunov function is selected and a stabilization controller is designed using **Theorem 1**. An appropriate adaptive law is also used to estimate the unknown parameters. The results of this section are proved by the fractional-order extension of Lyapunov direct method.

The coordinate transformations are defined below

$$\begin{cases} z_1 = x_1 \\ z_i = x_i - \alpha_{i-1} \end{cases}, 2 \leq i \leq n \,. \tag{12}$$

The result is shown as the following theorem:

Theorem 2. For the fractional-order system (10) with unknown parameters, if the adaptive backstepping control is designed as

$$u(t) = -f(x,t) - \hat{\boldsymbol{\delta}}^T F(x,t) + D^q\alpha_{n-1} - z_{n-1} - \gamma_n z_n - (\hat{\sigma} + \hat{\vartheta})sgn(z_n), \tag{13}$$

and adaptive laws of $\hat{\boldsymbol{\delta}}$, $\hat{\sigma}$ and $\hat{\vartheta}$ as

$$D^q\hat{\boldsymbol{\delta}} = z_n F(x,t), \tag{14}$$
$$D^q\hat{\sigma} = |z_n|, \tag{15}$$
$$D^q\hat{\vartheta} = |z_n|, \tag{16}$$

then states of system are asymptotic stabilization, i.e. $\lim_{t\to\infty}\|x(t)\| = 0$.

Proof. Step 1: Let $z_1 = x_1$ and differentiating two sides of it, one obtains

$$D^q z_1 = D^q x_1 = x_2 \,. \tag{17}$$

Suppose for the first subsystem, x_2 is the controller and α_1 is the virtual controller. Consider z_2 as the difference between the two controllers, i.e. $z_2 = x_2 - \alpha_1 \Rightarrow x_2 = z_2 + \alpha_1$. Now equation (17) is rewritten below

$$D^q z_1 = z_2 + \alpha_1 \,. \tag{18}$$

The Lyapunov function candidate V_1 is selected for subsystem (18) below

$$V_1 = \tfrac{1}{2} z_1^2 \,. \tag{19}$$

Taking the fractional-order derivative of Lyapunov function and utilizing the **Lemmas 1** and **2**, one has

$$D^q V_1 \leq z_1 D^q z_1 = z_1(z_2 + \alpha_1). \tag{20}$$

In order to fulfill the stability criterion stated in the previous section, the term α_1 is defined below

$$\alpha_1 = -\gamma_1 z_1 \,, \tag{21}$$

which leads to $D^q V_1 \leq -\gamma_1 z_1^2 + z_1 z_2$.

The term $z_1 z_2$ should be deleted in the above statement. By choosing α_1 in (21) and using **Theorem 1**, it is guaranteed that z_1 tends to zero, leading to subsystem stability (18).

Step 2: for the second subsystem, it yields

$$D^q z_2 = x_3 - D^q\alpha_1 \,. \tag{22}$$

Let us define $z_3 = x_3 - \alpha_2$ which gives $x_3 = z_3 + \alpha_2$, where x_3 is the controller and α_2 is the virtual controller.

By the above definition, Equation (22) is written below

$$D^q z_2 = z_3 + \alpha_2 - D^q\alpha_1 \,. \tag{23}$$

Consider the candidate of Lyapunov function below

$$V_2 = V_1 + \tfrac{1}{2} z_2^2 \,. \tag{24}$$

Tacking fractional-order differentiating V_2 in (24), it yields

$$D^q V_2 \leq D^q V_1 + z_2 D^q z_2 \leq -\gamma_1 z_1^2 + z_1 z_2 + z_2(z_3 + \alpha_2 - D^q\alpha_1). \tag{25}$$

The virtual control signal α_2 is defined below

$$\alpha_2 = -\gamma_2 z_2 - z_1 + D^q\alpha_1 \,. \tag{26}$$

By substituting (26) into (25), we get $D^q V_2 \leq -\gamma_1 z_1^2 - \gamma_2 z_2^2 + z_2 z_3$.

Similarly, in order to ensure the stabilization of subsystems in this stage, the term $z_2 z_3$ should be eliminated. Using the same method as above, we can go to step (n-1). Here the virtual controller α_{n-1} is selected as follows

$$\alpha_{n-1} = -\gamma_{n-1} z_{n-1} - z_{n-2} + D^q\alpha_{n-2} \,. \tag{27}$$

Finally, in step n the Lyapunov function is chosen below

$$V_n = V_{n-1} + \tfrac{1}{2} z_n^2 + \tfrac{1}{2}(\sigma - \hat{\sigma})^2 + \tfrac{1}{2}\left(\vartheta - \hat{\vartheta}\right)^2 + \tfrac{1}{2}\|\boldsymbol{\delta} -$$

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$\hat{\boldsymbol{\delta}}\|_2^2$,

$D^q V_n \leq D^q V_{n-1} + z_n D^q z_n - (\sigma - \hat{\sigma}) D^q \hat{\sigma} - (\vartheta - \hat{\vartheta}) D^q \hat{\vartheta} - (\boldsymbol{\delta} - \hat{\boldsymbol{\delta}})^T D^q \hat{\boldsymbol{\delta}} \leq -\gamma_1 z_1^2 - \gamma_2 z_2^2 \dots - \gamma_{n-1} z_{n-1}^2 + z_{n-1} z_n + z_n (f(x,t) + \boldsymbol{\delta}^T F(x,t) + \Delta f(x,t) + d(x,t) + u(t) - D^q \alpha_{n-1}) - (\sigma - \hat{\sigma}) D^q \hat{\sigma} - (\vartheta - \hat{\vartheta}) D^q \hat{\vartheta} - (\boldsymbol{\delta} - \hat{\boldsymbol{\delta}})^T D^q \hat{\boldsymbol{\delta}}$. (28)

By substituting the final controller (13) and the adaptive law (14) into (28), it is clear that

$$D^q V_n \leq -\sum_{i=1}^{n-1} \gamma_i z_i^2 + z_{n-1} z_n + z_n \Big((\boldsymbol{\delta} - \hat{\boldsymbol{\delta}})^T F(x,t) + \Delta f(x,t) + d(x,t) - z_{n-1} - \gamma_n z_n - (\hat{\sigma} + \hat{\vartheta}) sgn(z_n) \Big) - (\sigma - \hat{\sigma}) D^q \hat{\sigma} - (\vartheta - \hat{\vartheta}) D^q \hat{\vartheta} - z_n (\boldsymbol{\delta} - \hat{\boldsymbol{\delta}})^T F(x,t) \leq -\sum_{i=1}^{n} \gamma_i z_i^2 + |z_n| (|\Delta f(x,t)| + |d(x,t)|) - (\hat{\sigma} + \hat{\vartheta}) |z_n| - (\sigma - \hat{\sigma}) D^q \hat{\sigma} - (\vartheta - \hat{\vartheta}) D^q \hat{\vartheta} . \quad (29)$$

Utilizing **Assumptions 1** and **2**, and substituting the adaptive laws (15,16) into (29), one has

$$D^q V_n \leq -\sum_{i=1}^{n} \gamma_i z_i^2 + (\sigma + \vartheta) |z_n| - (\hat{\sigma} + \hat{\vartheta}) |z_n| - (\sigma - \hat{\sigma}) |z_n| - (\vartheta - \hat{\vartheta}) |z_n| . \quad (30)$$

It yields

$$D^q V_n \leq -\sum_{i=1}^{n} \gamma_i z_i^2 \leq 0 , \quad (31)$$

where $\gamma_i \geq 0$. So, according to **Theorem 1**, the suggested control strategy will guarantee that states of system (10) tend to zero. Thus, the **Theorem 2** is proved.

IV. Syncronization Of Fractional-Order Chaotic Systems Using Adaptive Backstepping Control

A. Problem Formulation

Consider the fractional-order chaotic system with uncertain parameters as below

$$\begin{cases} D^q x_i = x_{i+1} , \ \ 1 \leq i \leq n-1 \\ D^q x_n = f(x,t) + \boldsymbol{\delta}^T F(x,t) + \Delta f(x,t) + d^m(x,t) \end{cases} . \quad (32)$$

System (32) is a master system, where $q \in (0,1)$, $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in R^n$ is the state vector of the first system, $f(x,t) \in R$ and $F(x,t) \in R^{1\times p}$ are known nonlinear functions. $\boldsymbol{\delta} \in R^p$ is the uncertain parameter vector. $\Delta f(x,t)$ and $d^m(x,t)$ indicate uncertainty and external disturbance of the first system, respectively. It can be assumed that $\Delta f(x,t)$ and $d^m(x,t)$ are bounded by some positive constants i.e. $|\Delta f(x,t)| < \Delta_1$ and $|d^m(x,t)| < d_1$.

The slave system with the control signal $u(t) \in R$ is described below

$$\begin{cases} D^q y_i = y_{i+1} , \ \ 1 \leq i \leq n-1 \\ D^q y_n = g(y,t) + \boldsymbol{\theta}^T G(y,t) + \Delta g(y,t) + d^s(y,t) + u(t) \end{cases} , \quad (33)$$

where $y(t) = [y_1(t), y_2(t), \dots, y_n(t)]^T \in R^n$ is the state vector, $g(y,t)$ and $G(y,t) \in R^{1\times p}$ are known nonlinear functions, $\boldsymbol{\theta} \in R^p$ is the uncertain parameter vector and finally $\Delta g(y,t)$ and $d^s(y,t)$ indicate uncertainty and external disturbance of the slave system, respectively. It can be assumed that $\Delta g(y,t)$ and $d^s(y,t)$ are bounded by some positive constants i.e. $|\Delta g(y,t)| < \Delta_2$ and $|d^s(y,t)| < d_2$.

Assumption 3. With the above discussion, one can obtain that

$$|\Delta g(y,t) - \Delta f(x,t)| < \sigma_1 . \quad (34)$$
$$|d^s(y,t) - d^m(x,t)| < \vartheta_1 . \quad (35)$$

Assumption 4. The constants σ_1 *and* ϑ_1 are unknown positive.

Definition 5. To achieve the synchronization issue, the error among master and slave systems is defined as $e_i(t) = y_i(t) - x_i(t), i = 1,2, \dots, n$. So, the aim is to synchronize system (32) with system (33) via suggested control strategy, i.e. [16]

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y(t) - x(t)\| = 0 . \quad (36)$$

Subtracting (32) from (33), the synchronization error dynamics will be below

$$\begin{cases} D^q e_i = e_{i+1} , \ \ 1 \leq i \leq \\ D^q e_n = g(y,t) + \boldsymbol{\theta}^T G(y,t) + \Delta g(y,t) + d^s(y,t) - \\ \ \ f(x,t) - \boldsymbol{\delta}^T F(x,t) - \Delta f(x,t) - d^m(x,t) + u(t) \end{cases} \quad (37)$$

Clearly, the synchronization problem has turned into the stabilization issue of error system. To develop the controller, the backstepping method, which is described in the next section is used.

B. Controller Design

Here, adaptive backstepping control is utilized for synchronization of two different fractional-order chaotic systems with uncertainties, disturbances and unknown parameters. Coordinate transformations are defined as

$$\begin{cases} z_1 = e_1 \\ z_i = e_i - \alpha_{i-1} \end{cases} , 2 \leq i \leq n . \quad (38)$$

In the subsequent lines, the method of designing the controller is illustrated by the following theorem.

Theorem 3. By using the controller (39) and the adaptive laws (39-43) as follow

$$u(t) = -g(y,t) - \hat{\boldsymbol{\theta}}^T G(y,t) + f(x,t) + \hat{\boldsymbol{\delta}}^T F(x,t) + D^q \alpha_{n-1} - z_{n-1} - \gamma_n z_n - (\hat{\sigma}_1 + \hat{\vartheta}_1) sgn(z_n), \quad (39)$$
$$D^q \hat{\boldsymbol{\theta}} = z_n F(x,t), \quad (40)$$
$$D^q \hat{\boldsymbol{\delta}} = -z_n F(x,t), \quad (41)$$

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$$D^q\hat{\sigma}_1 = |z_n|, \quad (42)$$
$$D^q\hat{\vartheta}_1 = |z_n|, \quad (43)$$

the synchronization error moves toward zero, i.e. the slave system trajectories (33) tend to the master system trajectory (32).

Proof. Step 1: Let $z_1 = e_1$ and its derivative is

$$D^q z_1 = D^q e_1 = e_2\,, \quad (44)$$

where e_2 is the controller, α_1 is the virtual controller and z_1 is the difference between the two controllers, i.e. $z_2 = e_2 - \alpha_1 \Rightarrow e_2 = z_2 + \alpha_1$.

Therefore, equation (44) is rewritten below

$$D^q z_1 = z_2 + \alpha_1\,. \quad (45)$$

The Lyapunov function V_1 can be selected as $V_1 = \frac{1}{2}z_1^2$.

Taking the fractional-order derivative of V_1 and utilizing the **Lemmas 1** and **2**, it yields

$$D^q V_1 \le z_1 D^q z_1 = z_1(z_2 + \alpha_1). \quad (46)$$

The term α_1 is selected in a way to meet the stability criterion,

$$\alpha_1 = -\gamma_1 z_1\,, \quad (47)$$

which leads to $D^q V_1 \le -\gamma_1 z_1^2 + z_1 z_2$.

Step 2: Similarly, for the second subsystem, it yields

$$D^q z_2 = e_3 - D^q\alpha_1\,. \quad (48)$$

Let us define $z_3 = e_3 - \alpha_2$ which gives $e_3 = z_3 + \alpha_2$, where e_3 is the controller and α_2 is the virtual controller. Now one can get

$$D^q z_2 = z_3 + \alpha_2 - D^q\alpha_1\,. \quad (49)$$

The Lyapunov function is selected for the second subsystem as $V_2 = V_1 + \frac{1}{2}z_2^2$.

Tacking fractional-order derivative of V_2, one has

$$D^q V_2 \le D^q V_1 + z_2 D^q z_2 \le -\gamma_1 z_1^2 + z_1 z_2 + z_2(z_3 + \alpha_2 - D^q\alpha_1). \quad (50)$$

The virtual control signal α_2 is defined below

$$\alpha_2 = -\gamma_2 z_2 - z_1 + D^q\alpha_1\,. \quad (51)$$

Substituting (51) into (50), we get $D^q V_2 \le -\gamma_1 z_1^2 - \gamma_2 z_2^2 + z_2 z_3$.

Similarly, in order to ensure the stabilization of subsystems in this stage, the term $z_2 z_3$ should be eliminated. Using the same method as above, we can go to step (n-1). The virtual controller α_{n-1} is selected below

$$\alpha_{n-1} = -\gamma_{n-1} z_{n-1} - z_{n-2} + D^q\alpha_{n-2}\,. \quad (52)$$

Step n: The Lyapunov function is defined by

$$V_n = V_{n-1} + \frac{1}{2}z_n^2 + \frac{1}{2}(\sigma_1 - \hat{\sigma}_1)^2 + \frac{1}{2}(\vartheta_1 - \hat{\vartheta}_1)^2 + \frac{1}{2}\left\|\boldsymbol{\delta} - \widehat{\boldsymbol{\delta}}\right\|_2^2 + \frac{1}{2}\left\|\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}}\right\|_2^2,$$

$$D^q V_n \le D^q V_{n-1} + z_n D^q z_n - (\sigma_1 - \hat{\sigma}_1)D^q\hat{\sigma}_1 - (\vartheta_1 - \hat{\vartheta}_1)D^q\hat{\vartheta}_1 - (\boldsymbol{\delta} - \widehat{\boldsymbol{\delta}})^T D^q\widehat{\boldsymbol{\delta}} - (\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}})^T D^q\widehat{\boldsymbol{\theta}} \le -\gamma_1 z_1^2 - \gamma_2 z_2^2 \ldots - \gamma_{n-1} z_{n-1}^2 + z_{n-1} z_n + z_n\big(g(y,t) + \boldsymbol{\theta}^T G(y,t) + \Delta g(y,t) + d^s(y,t) - f(x,t) - \boldsymbol{\delta}^T F(x,t) - \Delta f(x,t) - d^m(x,t) + u(t) - D^q\alpha_{n-1}\big) - (\sigma_1 - \hat{\sigma}_1)D^q\hat{\sigma}_1 - (\vartheta_1 - \hat{\vartheta}_1)D^q\hat{\vartheta}_1 - (\boldsymbol{\delta} - \widehat{\boldsymbol{\delta}})^T D^q\widehat{\boldsymbol{\delta}} - (\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}})^T D^q\widehat{\boldsymbol{\theta}}\,. \quad (53)$$

By substituting the final controller (39) and adaptive laws (40,41) into (53)

$$D^q V_n \le -\sum_{i=1}^{n-1}\gamma_i z_i^2 + z_{n-1} z_n + z_n\Big(\boldsymbol{\theta}^T G(y,t) + \Delta g(y,t) + d^s(y,t) - \boldsymbol{\delta}^T F(x,t) - \Delta f(x,t) - d^m(x,t) - \widehat{\boldsymbol{\theta}}^T G(y,t) + \widehat{\boldsymbol{\delta}}^T F(x,t) + D^q\alpha_{n-1} - z_{n-1} - \gamma_n z_n - (\hat{\sigma}_1 + \hat{\vartheta}_1)sgn(z_n)\Big) - (\sigma_1 - \hat{\sigma}_1)D^q\hat{\sigma}_1 - (\vartheta_1 - \hat{\vartheta}_1)D^q\hat{\vartheta}_1 + z_n(\boldsymbol{\delta} - \widehat{\boldsymbol{\delta}})^T F(x,t) - z_n(\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}})^T G(y,t). \quad (54)$$

It is clear

$$\Rightarrow D^q V_n \le -\sum_{i=1}^{n}\gamma_i z_i^2 + |z_n|(|\Delta g(y,t) - \Delta f(x,t)| + |d^s(y,t) - d^m(x,t)|) - (\hat{\sigma}_1 + \hat{\vartheta}_1)|z_n| - (\sigma_1 - \hat{\sigma}_1)D^q\hat{\sigma}_1 - (\vartheta_1 - \hat{\vartheta}_1)D^q\hat{\vartheta}_1\,. \quad (55)$$

Utilizing **Assumptions 3** and **4**, and by substituting the adaptive laws (42,43) into (55)

$$D^q V_n \le -\sum_{i=1}^{n}\gamma_i z_i^2 + (\sigma_1 + \vartheta_1)|z_n| - (\hat{\sigma}_1 + \hat{\vartheta}_1)|z_n| - (\sigma_1 - \hat{\sigma}_1)|z_n| - (\vartheta_1 - \hat{\vartheta}_1)|z_n| \le -\sum_{i=1}^{n}\gamma_i z_i^2 \le 0, \quad (56)$$

where $\gamma_i \ge 0$. Then, using the Lyapunov stability **Theorem 1**, it can be concluded that the synchronization error moves toward zero and the synchronization is realized. Thus, the **Theorem 3** is proved.

V. Numerical Example

Here, three examples are represented to investigate the usefulness of the developed method. Numerical simulations have been implemented with MATLAB software.

A. Example1

In this example, the suggested controller (13) and the adaptive laws (14-16) are used to stabilize the uncertain fractional-order Genesio-Tesi system. The fractional-order Genesio-Tesi system with unknown parameters is given by the following equation

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$$\begin{cases} D^q x_1 = x_2 \\ D^q x_2 = x_3 \\ D^q x_3 = -\delta_1 x_1 - \delta_2 x_2 - \delta_3 x_3 + \delta_4 x_1^2 \end{cases} . \quad (57)$$

Fig. 1 shows the chaotic motion of the fractional-order Genesio-Tesi system (57) for $\delta_1 = 6$, $\delta_2 = 2.92$, $\delta_3 = 1.2$ and $\delta_4 = 1$. Initial conditions of the system (57) are chosen as $x_1(0) = -0.1$ $x_2(0) = 0.5$ and $x_3(0) = 0.2$. The fractional-order q is also selected to 0.94.

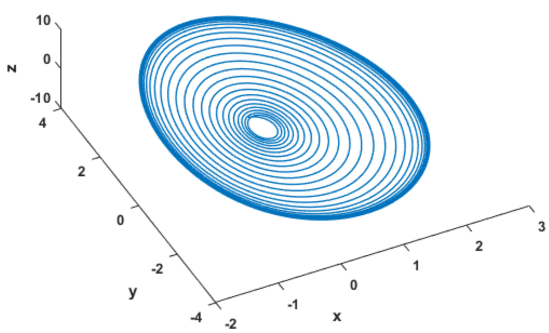

Fig. 1. The chaotic trajectory of the fractional-order Genesio-Tesi system (57).

The design parameters are selected as $\gamma_1 = 4$, $\gamma_2 = 3$, $\gamma_1 = 2.5$ and the order of derivative is $q = 0.94$. The uncertainties and external disturbances are selected below

$$\begin{cases} D^q x_1 = x_2 \\ D^q x_2 = x_3 \\ D^q x_3 = -\delta_1 x_1 - \delta_2 x_2 - \delta_3 x_3 + \delta_4 x_1^2 \end{cases} . \quad (58)$$

The control strategy in **Theorem 2** is utilized to stabilize the system (57) with unknown parameters, uncertainties and disturbances.

Figs. 2-4 illustrate the state trajectories of the system (57), the time evolutions of the update vector parameter $\hat{\boldsymbol{\delta}}$ and the time history of the control input (13), respectively. As it can be seen, the state trajectories of the system (57) tend to zero and the control signal is practical. This means that the proposed backstepping controller can be used to stabilize fractional-order Genesio-Tesi system. Also, the system unknown parameters are fully estimated and converge to a constant value.

To compare the performance of our proposed method, the fractional-order sliding mode control strategy presented in [28] has been simulated to stabilize fractional-order Genesio-Tesi system. Fig. 5 shows the time response of the control signal via the proposed fractional-order sliding mode control method in [28]. As it is obvious, the control signal has permanent chattering, which limits the practical performance of the proposed controller in [28]. These results confirm the superiority of the proposed adaptive backstepping method to stabilize the fractional-order Genesio-Tesi system.

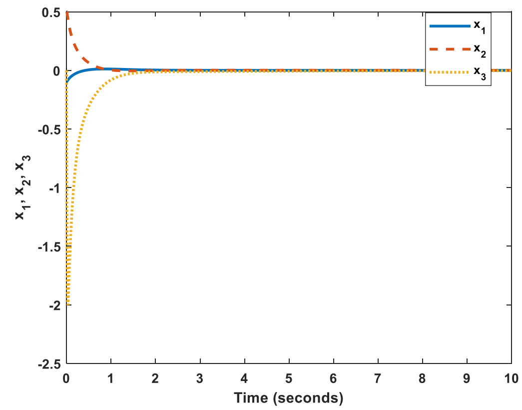

Fig. 2. State trajectories of the fractional-order Genesio–Tesi system controlled with (13).

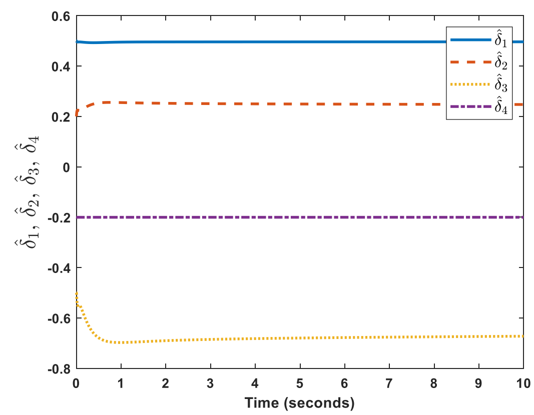

Fig. 3. Time response of the adaptive vector parameter $\hat{\boldsymbol{\delta}}$.

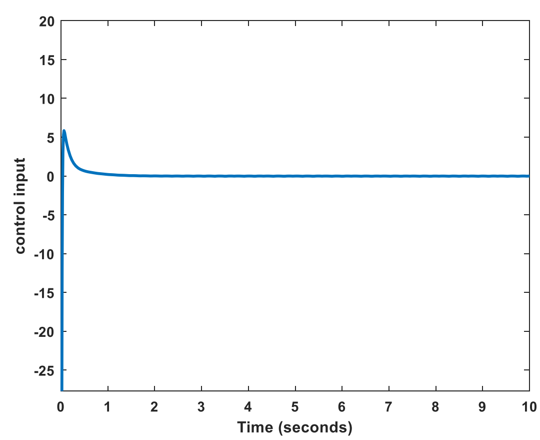

Fig. 4. Time history of the control input (13) applied to the fractional-order Genesio–Tesi system.

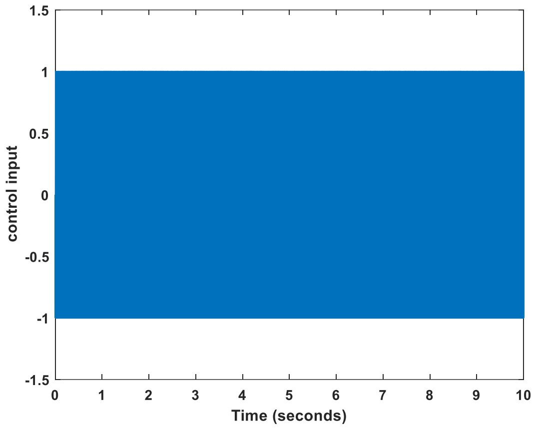

Fig. 5. The time response of the applied control input via the method in Ref. [28].

B. Example 2

In this example, the suggested adaptive backstepping controller (39) is utilized to synchronize two different uncertain fractional-order Duffing–Holmes system and fractional-order Gyros system. The fractional-order Duffing–Holmes system with unknown parameter is given by the following equation,

master system:

$$\begin{cases} D^q x_1 = x_2 \\ D^q x_2 = x_1 - \delta_1 x_2 - x_1^3 + 0.3\cos\ t \end{cases}, \quad (59)$$

where δ_1= 0.25. Fig. 6 shows the chaotic behavior of the fractional-order Duffing–Holmes system (59) for $q =$ 0.98 and the initial condition $[x_1(0), x_2(0)]^T = [0.3, -0.2]^T$.

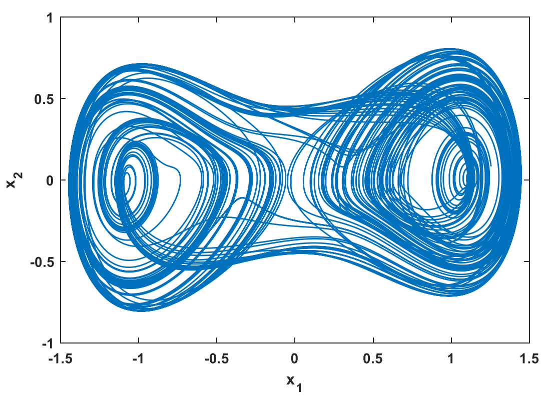

Fig. 6. The chaotic trajectory of the fractional-order Duffing–Holmes system (59).

The fractional-order Gyros system is given below,

slave system:

$$\begin{cases} D^q y_1 = y_2 \\ D^q y_2 = -100\frac{(1-cos y_1)^2}{sin^3 y_1} - \sin y_1 - \theta_1 y_1 - \\ \quad \theta_2 y_2^3 + (1 + 35 sin\omega t)\sin y_1 + u(t) \end{cases} \quad (60)$$

where $\theta_1 = 0.5$, $\theta_2 = 0.05$ and $\omega = 1.8$. Chaotic attractor of this system is revealed in Fig. 7 for q = 0.98 and the initial condition $[y_1(0), y_2(0)]^T = [-0.1, 0.2]^T$.

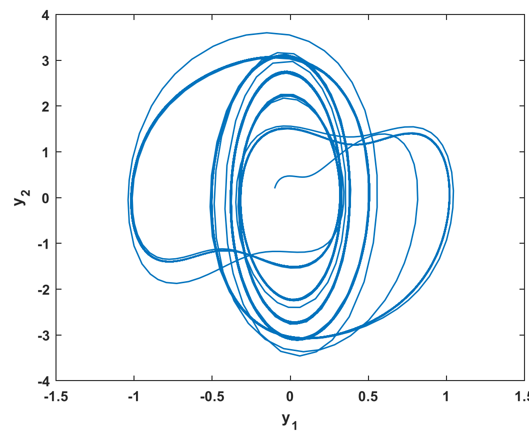

Fig. 7. The chaotic trajectory of the fractional-order Gyros system (59).

The design parameters are selected as $\gamma_1 = 2$, $\gamma_2 = 2$ and the order of derivative is $q = 0.98$. Furthermore, the uncertainties and external disturbances are selected below

$$\Delta f(x,t) = 0.1\sin(x_2)\cos(x_1), \quad d^m(t) = 0.1\cos(3t). \quad (61)$$

$$\Delta g(y,t) = 0.4\sin(y_1)\sin(y_2), \ d(t) = 0.2\cos(t). \quad (62)$$

Therefore, the control algorithm in **Theorem 3** is used to synchronize system (57) and system (59). Figs. 8-10 demonstrate the trajectory of master and slave systems, synchronization error and the time evolutions of the update vector parameters $\hat{\boldsymbol{\delta}}$ and $\hat{\boldsymbol{\theta}}$, when the controller (39) is utilized. The time history of the control input (39) is shown in Fig. 11. It is clear that the proposed controller has been able to synchronize both master and slave systems even in the presence of uncertainties and disturbances. Also unknown parameters of the system converge to constant value. To compare the results, the fractional-order sliding mode control presented in [29] is simulated for the two systems (59) and (60).The control signal in [29] is shown in Fig. 12. Clearly, the control signal in [29] has fluctuations. These results confirm the superiority of our proposed adaptive backstepping method in synchronizing two systems (59) and (60).

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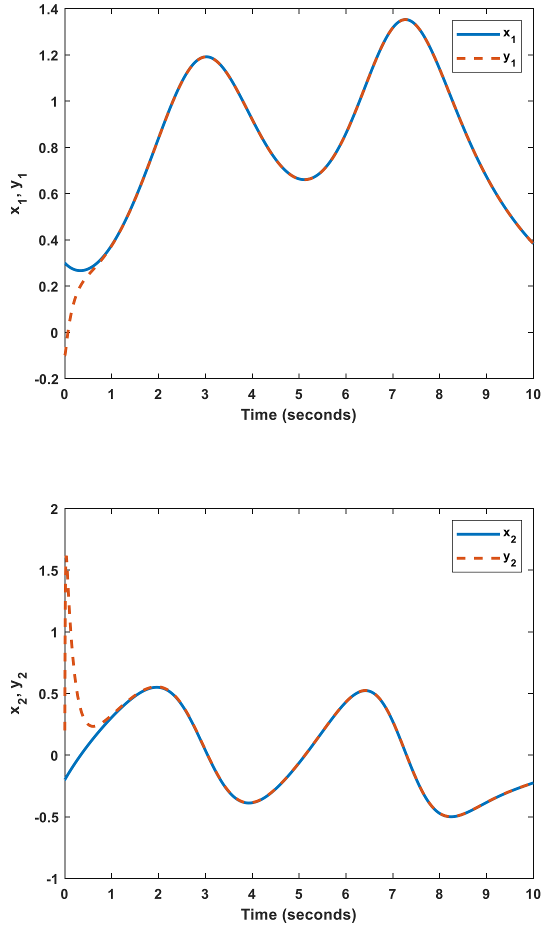


Fig. 8. State trajectories of the fractional-order Duffing–Holmes and fractional-order Gyros systems.

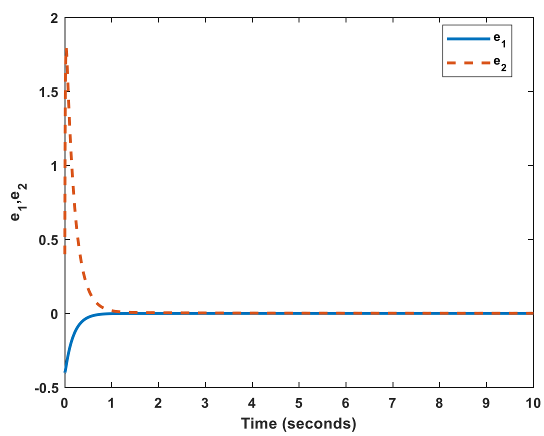


Fig. 9. Synchronization errors of the fractional-order Duffing–Holmes and fractional-order Gyros systems.

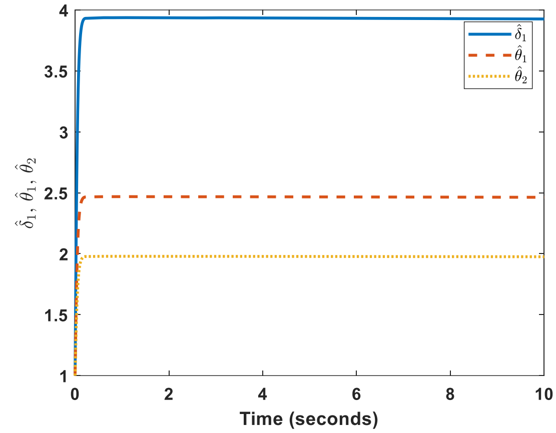


Fig. 10. Time response of the adaptive vector parameters $\hat{\boldsymbol{\delta}}$ and $\hat{\boldsymbol{\theta}}$.

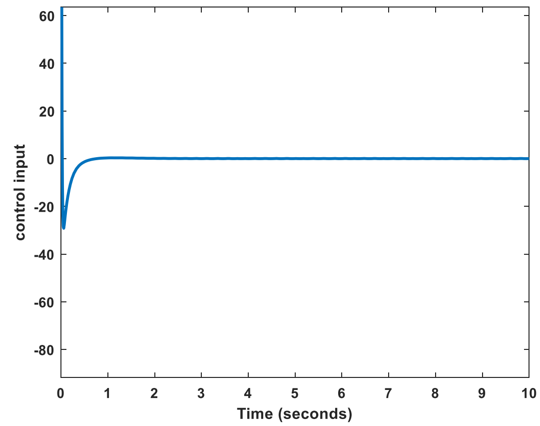


Fig. 11. Time history of the control input (39).

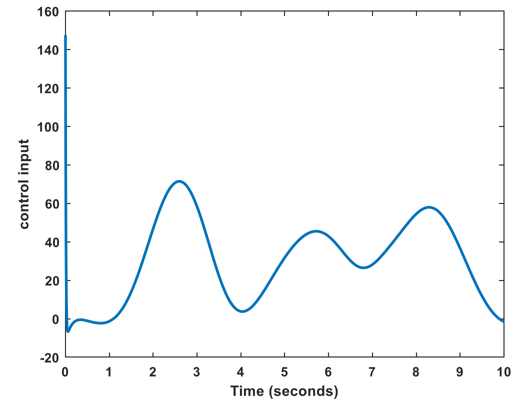


Fig. 12. The time response of the applied control input via the method in Ref. [29].

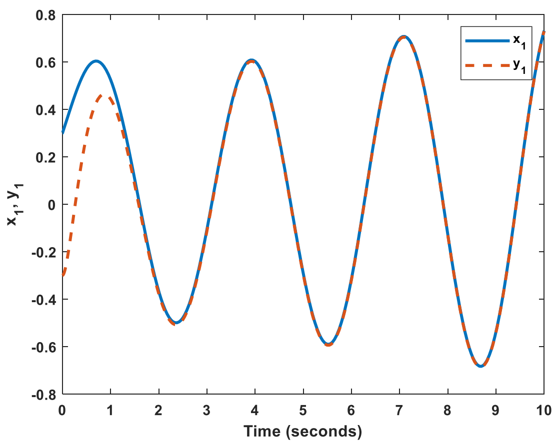
C. Example 3

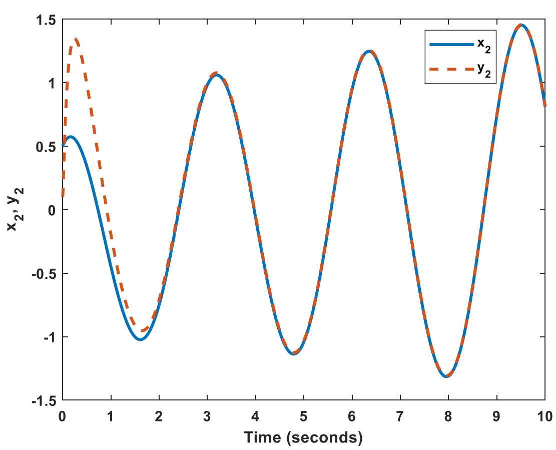
In this example, the synchronization of two similar indeterminate fractional-order Genesio-Tesi systems is examined. The master system is considered as in (57). The slave system can be given below

$$\begin{cases} D^q y_1 = y_2 \\ D^q y_2 = y_3 \\ D^q y_3 = -\theta_1 y_1 - \theta_2 y_2 - \theta_3 y_3 + \theta_4 y_1^2 \end{cases}, \qquad (63)$$

where $\theta_1 = 6$, $\theta_2 = 2.92$, $\theta_3 = 1.2$ and $\theta_4 = 1$. Initial conditions of the system (63) are chosen as $y_1(0) = -0.3$ $y_2(0) = 0.1$ and $y_3(0) = 1.8$. The fractional-order q is also selected to 0.94.

Figs. 13-15 reveal the trajectories of the master and slave systems, the synchronization error and the estimate of the master and slave system parameters in the presence of suggested controller (39). It is obvious that the synchronization errors quickly reach zero and accordingly it can be said that both master and slave systems are synchronized. Fig. 16 shows the time response of control input (39). Fig. 17 shows the control input presented in [30]. It is clear that the control input in [30] has large fluctuations while the proposed control signal in our paper lacks chattering and is practical.





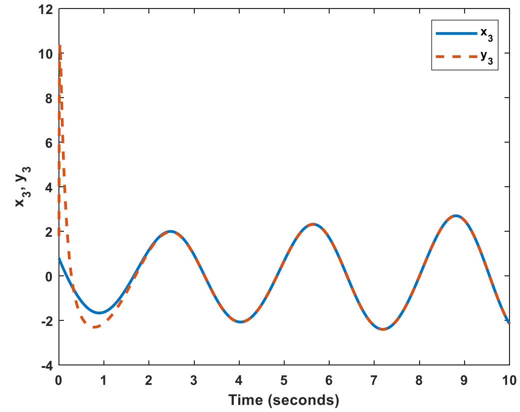


Fig. 13. State trajectories of two fractional-order Genesio–Tesi systems.

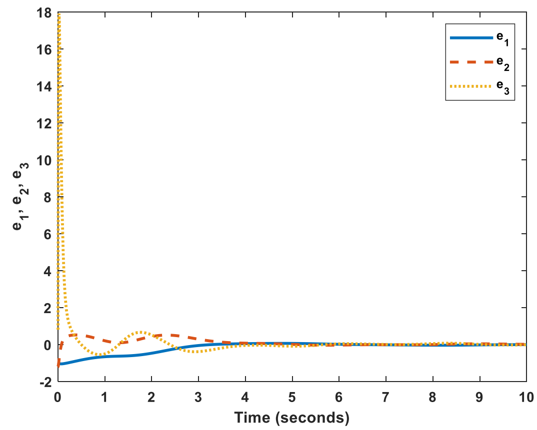


Fig. 14. Synchronization errors of two fractional-order Genesio–Tesi systems.

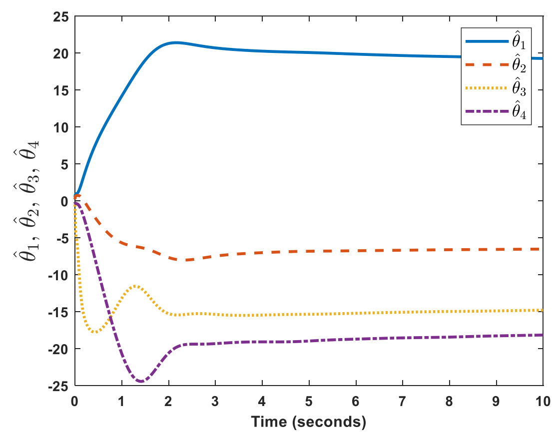


Fig. 15. Time response of the adaptive vector parameter $\hat{\boldsymbol{\delta}}$.

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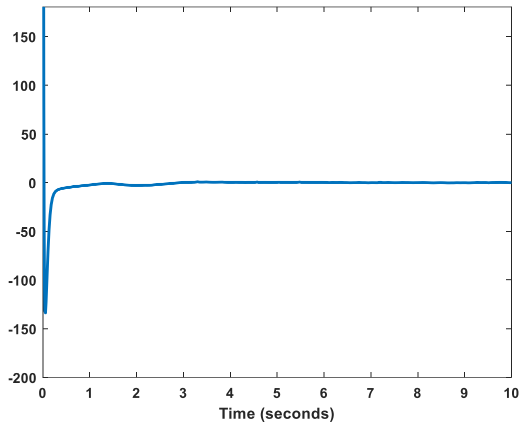

Fig. 16. Time history of the control input (39).

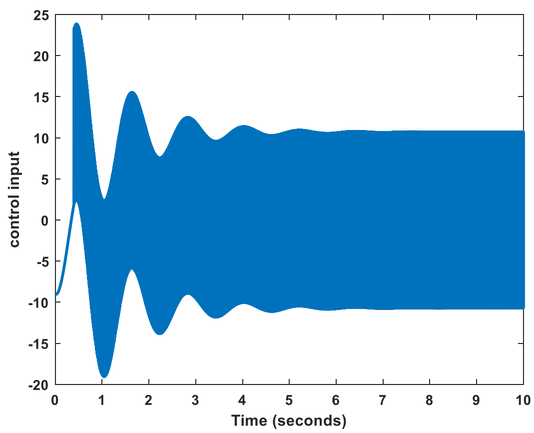

Fig. 17. The time response of the applied control input via the method in Ref. [30].

VI. Conclusion

In this work, a fractional controller utilizing adaptive backstepping technique is suggested. This controller can be used for a general class of systems. At the first, assuming the presence of uncertainties and disturbances, the stabilization of fractional-order chaotic system with unknown parameters is investigated. Following the discussion, the synchronization of two different fractional-order chaotic systems with uncertain parameters is studied. The detailed analysis is pursuant to fractional Lyapunov theorem and adaptive laws to certify the stability of the controlled systems. It is needed to mention that the suggested method is simple and practical. Three examples are presented to examine the effectiveness of suggested approach and provide a deeper view of the proposed controller applications. The results of our suggested technique are compared with the results of some available papers. The simulation results show better performance of the proposed controller in this paper.

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