Stability Analysis of Switched Time-Delay Systems; a Multiple Case

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***Abstract*—This paper investigates the problem of exponential stability of switched systems with multiple time-varying delays. By using the multiple discontinuous Lyapunov function (MDLF) approach and the mode-dependent average dwell time (MDADT) switching signal, new conditions in the form of linear matrix inequalities (LMIs) are proposed, which is the first attempt in this area. These conditions guarantee the exponential stability of the switched system with stable and unstable subsystems where multiple time-varying delays are considered in the states. In this method all subsystems cannot be unstable, and switched system should compose of at least one stable subsystem. Furthermore, two examples are given to illustrate the effectiveness of the obtained theoretical results.**

***Keywords*: Stability Analysis, Switched Systems, Multiple Time Delay, Multiple Lyapunov Function.**

I. INTRODUCTION

Switched systems, as a significant class of hybrid systems, have been widely studied in recent years [1-3]. It is known that many real-world systems can be modeled as switched systems, for example, multi-agent systems [4], control in networks [5, 6], chemical process systems [7], flight control systems [8], and communication systems [9]. Mainly in continuous-time systems, the stability analysis tools used by Lyapunov include the common Lyapunov function method (CLF) and the multiple Lyapunov function(MLF) method [10]. Since it is difficult to find a common Lyapunov function for all subsystems, it is impossible to permanently ensure the stability of a switched system under an arbitrary switching signal. Therefore, a logical solution is to consider stability with constrained switching signals. The literature confirms that the MLF approach not only has more flexibility but also it is possible to provide less conservatism in the analysis of switched systems [11]. Recently, the authors in [12], developed the MDLF approach for stability analysis with a new switching regime. In this work, fast switching applies to unstable subsystems, and slow switching applies to stable subsystems.

Research in the field of stability analysis with constrained switching have been chiefly motivated by state-dependent switching [13], time-dependent switching, and a combination of them. However, the time-dependent switching strategy has been recognized to be more flexible and efficient than state-dependent switching in the stability analysis [12]. Time-dependent switching signals usually consist of three significant groups of switching regimes; dwell time (DT), average dwell time (ADT), and MDADT switching signals. In [14] using DT, a sufficient condition to ensure the asymptotic stability of switched systems with all unstable subsystems has been proposed. Stability analysis for switched positive linear systems with average dwell time switching is revisited and discussed in both continuous-time and discrete-time contexts in [15]. The MDADT approach is more suitable in practice than the DT and ADT methods because the switching law allows each subsystem to have its own separate ADT [16]. Some results on the MDADT approach applied to switched systems are presented in [17-20]. Here, each Lyapunov function is piecewise continuous throughout the DT for an activated system mode. Researchers have shown that the MDLF approach can achieve tighter bounds on ADT or MDADT.

On the other hand, time delays widely exist in many practical systems, such as aircraft, chemical or process control systems, and communication networks, either in the state, the control input, or the measurements [15]. Hence, the study of time-delay systems has been of great interest in many branches of science and engineering applications. For instance, in [21], a new delay-dependent robust stability condition for a type of uncertain linear system is proposed using the augmented Lyapunov–Krasovskii functional and Jensen's inequality. Moreover, it is worth to mention that the multiple time-varying delay attract considerable research interest in recent years (see [22-25] and references therein). In [26], the delay-dependent stability condition of Lurie control systems with multiple time delays is obtained by applying the technique of analyzing inequality and the method of decomposing the matrices. Switched time delay systems have strong engineering contextual in power systems and multi-rate control systems [27, 28]. So, we make a try to introduced multiple time-varying delays to switched systems. It is exciting and challenging to investigate the stability problem of switched time-delay systems. In [29], the authors have presented the model reference tracking control problem for a class of switched nonlinear systems with multiple time-varying delays. From the studies conducted, we can mention [30-33] in singular systems, which provides the stability analysis of a class of singular switched delayed systems with arbitrary switching signals.

In [34], the authors obtained stability and stabilization conditions for switched time delay systems, but the intended delays for the time delay system are not multiple, which somehow limits the application. This paper extends the time delays considered in [34] to multiple time-varying delays for switched system stability, which removes the previous limitation.

In this paper, we focus on the problem of stability analysis of a continuous-time switched system, including stable and unstable subsystems and multiple time-varying delays in the states. Using the MDLF approach and the MDADT switching, a set of sufficient conditions is proposed to guarantee the exponential stability of the switched system. Compared with the relevant literature, the main contribution of this paper is as follows: (i) The stability issue of switched systems with multiple time-varying delays is studied by employing the MDLF approach, which is the first attempt in this area; (ii) New sufficient criteria are derived, ensuring the stability of switched systems with multiple time-varying delays. This offers a tighter dwell time-bound with less conservativeness. (iii) Typically, in the literature, subsystems are only considered stable, which is a restriction. However, in this paper, subsystems can be both stable and unstable. The rest of this paper is organized as follows. Some necessary concepts of switched delay systems are reviewed in Section 2. In Section 3, exponential stability conditions for switched systems with multiple time-varying delays are established. Two examples show the effectiveness of the obtained results in Section 4, and we conclude this paper in Section 5.

II. System Description and Preliminaries

Consider a class of switched systems with multiple time-varying delays as the following equation:

$$\dot{x}(t) = A_0(\beta)x(t) + \sum_{j=1}^{m} A_j(\beta)x\left(t - d_j(t)\right), \qquad (1)$$

$$x(t) = \phi(t)\,, t \in [-d, 0],$$

where $x(t) \in \mathbb{R}^n$, denotes the state, and $\{(A_0(\beta), \dots, A_m(\beta)) : \beta \in \mathcal{S}\}$ is a set of matrices, which take its values by an index set $\mathcal{S} = \{1,2,\dots,N\}$, in which $N > 1$ represents the number of subsystems. In here, $\mathcal{S} = \mathcal{G} \cup \mathcal{U}$, where $\mathcal{G}$ and $\mathcal{U}$ signify the sets composed of stable and unstable subsystems respectively, and $\beta(t) : [0, \infty) \to \mathcal{S}$ denotes a switching signal which is a piecewise constant function of time t that might depend on t or $x(t)$. From now on, for simplicity, at a specified time t, the $\beta(t)$, denoted by β. $d_j(t)$, $1 \le j \le m$, represents the time-varying delays satisfying (A1) as stated below:

(A1) $0 \le d_j(t) \le d_j$, $\dot{d}_j(t) \le \sigma_j \le 1$,

let $d = \max_{1 \le j \le m} \{d_j(t)\}$. Furthermore, $\phi(t)$ is an initial vector valued continuous function on $[-d, 0]$ for a known constant $d > 0$. It should be emphasized from the theory of delay differential equations [35] that the existence of the solutions of a non-switched linear delay system is guaranteed by a continuous and piecewise differentiable initial condition. Consider the following non-switched delay system:

$$\dot{x}(t) = Ax(t) + A_d x(t - d),$$

$$x(\theta) = \phi(\theta)\,, \theta \in [-d, 0].$$

When $\phi(0)$ is continuous, then there exists a unique solution $x(\phi)$ defined on $[-d, \infty)$ that coincides with ϕ on $[-d, 0]$. By the Lagrange's formula, this solution is given by

$$x(t) = \exp^{At} x(0) + \int_0^t \exp^{A(t-\theta)} A_d x(\theta - d) d\theta$$

$$= \exp^{At} x(0) + \int_{-d}^{t-d} \exp^{A(t-\theta-d)} A_d x(\theta) d\theta.$$

This is carried over to linear switched-delay systems since the state does not experience any jump at the switching instants, based on [27].

In accordance with the switching signal β, we have the switching sequence

$\{(\beta_0, t_0), (\beta_1, t_1), (\beta_2, t_2), \cdots, (\beta_k, t_k), \cdots, |\beta_k \in \mathcal{S},\ k = 0,1,2,\cdots\}$ with $t_0 = 0$, which means that the β_k-th subsystem is activated when $t \in [t_k, t_{k+1})$.

Now, considering above systems, the following definitions and lemmas are made through the paper:

Definition 1 [11]: Switched system (1) is said to be exponentially stable under the switching signal $\beta(t)$ if there exist constants $\varepsilon > 0$ and $k > 0$ such that, for any compatible initial condition $\phi(t)$, the solution $x(t)$ to the switched system (1) satisfies $\|x(t)\| \le ke^{-\varepsilon(t-t_0)}\|x(t_0)\| \quad \forall t \ge t_0$.

Definition 2 [16]: For a switching signal $\beta(t)$ and any time interval $[t_1, t_2]$, $N_{\beta i}(t_1, t_2)$ represents the number of times that the ith subsystem is activated, and $T_i(t_1, t_2)$ represents the sum of the consecutively time of the ith subsystem, $i \in \mathcal{S}$. We say that $\beta(t)$ has a mode-dependent average dwell (MDADT) time τ_i^a if there exist constants N_{0i} and τ_i^a such that: $N_{\beta i}(t_1, t_2) \le N_{0i} + \frac{T_i(t_1,t_2)}{\tau_i^a}$, $\forall t_2 \ge t_1 \ge 0$.

Definition 3 [12]: For a switching signal $\beta(t)$ and any time interval $[t_1, t_2]$, $N_{\beta i}(t_1, t_2)$ represents the number of times that the ith subsystem is activated, and $T_i(t_1, t_2)$ represents the sum of the running time of the ith subsystem, $i \in \mathcal{S}$. If there exist constants N_{0i} and τ_i^a such that: $N_{\beta i}(t_1, t_2) \ge N_{0i} + \frac{T_i(t_1,t_2)}{\tau_i^a}$, $\forall t_2 \ge t_1 \ge 0$, then we say that the constant τ_{ai}, is the MDADT of the fast-switching signal $\beta(t)$.

For stability analysis, for any time interval $[t_k, t_{k+1})$ between two consecutive switching instances, we divide the interval into $G_{\beta(t_k)}$ divisions, and the length of each section is denoted by $H^i_{\beta(t_k)}$, $\forall i \in \{1, \dots, G_{\beta(t_k)}\}$ as stated in [12]. For this purpose, define the $J^i_{\beta(t_k)} = \sum_{j=1}^{i} H^j_{\beta(t_k)}$, where $J^0_{\beta(t_k)} = 0$, $\forall i \in \{0,1,\dots, G_{\beta(t_k)}\}$, and denote $L^i_{\beta(t_k)} = [t_k + J^i_{\beta(t_k)}, t_k + J^{i+1}_{\beta(t_k)})$, $i \in \mathcal{R}_{\beta(t_k)} = \{0,1,\dots, G_{\beta(t_k)} - 1\}$.

Then, the time interval $[t_k, t_{k+1})$ can be described as $[t_k, t_{k+1}) = \bigcup_i L^i_{\beta(t_k)}$, $i \in \mathcal{R}_{\beta(t_k)}$.

Lemma 1 [12]: consider the following switched system

$$\dot{x}(t) = f_{\beta(t)}(x(t)), \tag{2}$$

for $\beta \in \mathcal{G}$, if there exist scalars $\alpha_\beta > 0$, $0 < \eta_\beta \le 1$, $\mu_\beta > 1$, satisfying $(\eta_\beta)^{G_\beta - 1}\mu_\beta > 1$ and also for $\beta \in \mathcal{U}$, there exist scalars $\alpha_\beta < 0$,$0 < \eta_\beta \le 1$, $0 < \mu_\beta < 1$. If there exist a set of continuously differentiable non-negative functions $V_\beta{}^i(x(t))$, $\beta \in \mathcal{S}$, $i \in \mathcal{R}_{\beta(t_k)}$, and two class $\mathcal{K}_\infty$ functions κ_1 and κ_2, such that $\forall i \in \mathcal{R}_{\beta(t_k)}$,

$$\kappa_1(\|x(t)\|) \le V_\beta{}^i(x(t)) \le \kappa_2(\|x(t)\|)\ \forall \beta \in \mathcal{S},$$

$$\dot{V}_\beta{}^i(x(t)) + \alpha_\beta V_\beta{}^i(x(t)) \le 0\ \forall \beta \in \mathcal{S},$$

$$V_\beta{}^i\left(x\left(t_k + J^i_\beta\right)\right) - \eta_\beta V_\beta{}^{i-1}\left(x\left(t_k + J^i_\beta\right)\right) \le 0\ \forall \beta \in \mathcal{S}, i \ne 0$$

$$V_\beta{}^0(x(t)) - \mu_\beta V_\gamma{}^{G_\gamma - 1}(x(t)) \le 0\ \forall (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \ne \gamma,$$

$$V_\gamma{}^0(x(t)) - \mu_\gamma V_\beta{}^{G_\beta - 1}(x(t)) \le 0\ \forall (\beta, \gamma) \in \mathcal{G} \times \mathcal{U},$$

then switched system (2) is exponentially stable for any MDADT switching signals satisfying

$$\begin{cases} \tau_\beta^a \ge \tau_\beta^{a*} = \frac{\ln \mu_\beta + (G_\beta - 1)\ln \eta_\beta}{\alpha_\beta}, \beta \in \mathcal{G}, \\ \tau_\beta^a \le \tau_\beta^{a*} = \frac{\ln \mu_\beta + (G_\beta - 1)\ln \eta_\beta}{\alpha_\beta}, \beta \in \mathcal{U}. \end{cases} \tag{3}$$

According to Lemma 1, the switching law depends on the switching frequency to ensure system stability. The switching frequency in stable subsystems should be slowed down, so that the dwell time exceeds a specific value (this specific value is denoted by $\tau_\beta^{a*}$for $\beta \in \mathcal{G}$). Also, for unstable subsystems, the switching frequency must be speed up so that the dwell time is less than a specific value (this specific value is denoted by τ_β^{a*} for $\beta \in \mathcal{U}$). Slow switching and fast switching determine these two values, which were expressed in Definition 2 and Definition 3, respectively.

It should be noted that in this method, it is sufficient to know the stability and instability of the active subsystem, which is an advantage.

It is worth mentioning, since $(\eta_\beta)^{G_\beta - 1}\mu_\beta > 1$ for $\beta \in \mathcal{G}$, then $\ln(\eta_\beta)^{G_\beta - 1}\mu_\beta > 0$. So, we have $\ln \mu_\beta + (G_\beta - 1)\ln \eta_\beta > 0$. On the other hand, for $\beta \in \mathcal{G}$, $\alpha_\beta > 0$. So, $\tau_\beta^{a*} > 0$ for $\beta \in \mathcal{G}$.

For $\beta \in \mathcal{U}$, we have, $0 < \eta_\beta \le 1$, $0 < \mu_\beta < 1$. So, $\ln \mu_\beta < 0$ and $(G_\beta - 1)\ln \eta_\beta < 0$. Hence, we have, $\ln \mu_\beta + (G_\beta - 1)\ln \eta_\beta < 0$. On the other hand, for $\beta \in \mathcal{U}$, $\alpha_\beta < 0$. So finally, $\tau_\beta^{a*} > 0$ for $\beta \in \mathcal{U}$.

III. Stability Analysis

Now, we establish exponential stability conditions for the switched system (1) by using the mode-dependent average dwell time approach and the multiple discontinuous Lyapunov function method. The main results are given as the following theorems.

Theorem 1: Consider the switched system (1). Suppose (A1) holds and for given scalars $\alpha_\beta > 0$, $0 < \eta_\beta \le 1$, $\mu_\beta > 1$, $\beta \in \mathcal{G}$ satisfying $(\eta_\beta)^{G_\beta - 1}\mu_\beta > 1$, and $\alpha_\beta < 0$, $0 < \eta_\beta \le 1$, $0 < \mu_\beta < 1$, $\beta \in \mathcal{U}$ if there exist matrices $P^i_\beta > 0$, $Q^i_{j\beta} > 0$, $R^i_{j\beta} > 0$, $X^i_\beta, Y^i_\beta, T^i_\beta = \begin{pmatrix} T^i_{\beta 11} & T^i_{\beta 12} \\ T^i_{\beta 21} & T^i_{\beta 22} \end{pmatrix} \ge 0\ \forall \beta \in \mathcal{S}, i \in \mathcal{R}_{\beta(t_k)}$, such that

$$\begin{cases} P^i_\beta \le \eta_\beta P^{i-1}_\beta, & \beta \in \mathcal{S}, i \ne 0 \\ Q^i_{j\beta} \le \eta_\beta Q^{i-1}_{j\beta}, & \beta \in \mathcal{S}, i \ne 0 \\ R^i_{j\beta} \le \eta_\beta R^{i-1}_{j\beta}, & \beta \in \mathcal{S}, i \ne 0 \end{cases} \tag{4.a}$$

$$\begin{cases} P_\beta^0 \le \mu_\beta P_\gamma^{G_\gamma-1}, & (\beta,\gamma)\in\mathcal{G}\times\mathcal{S},\beta\ne\gamma \\ Q_{j\beta}^0 \le \mu_\beta Q_{j\gamma}^{G_\gamma-1}, & (\beta,\gamma)\in\mathcal{G}\times\mathcal{S},\beta\ne\gamma \\ R_{j\beta}^0 \le \mu_\beta R_{j\gamma}^{G_\gamma-1}, & (\beta,\gamma)\in\mathcal{G}\times\mathcal{S},\beta\ne\gamma \end{cases} \quad (4.b)$$

$$\begin{cases} P_\gamma^0 \le \mu_\gamma P_\beta^{G_\beta-1}, & (\beta,\gamma)\in\mathcal{G}\times\mathcal{U} \\ Q_{j\gamma}^0 \le \mu_\gamma Q_{j\beta}^{G_\beta-1}, & (\beta,\gamma)\in\mathcal{G}\times\mathcal{U} \\ R_{j\gamma}^0 \le \mu_\gamma R_{j\beta}^{G_\beta-1}, & (\beta,\gamma)\in\mathcal{G}\times\mathcal{U} \end{cases} \quad (4.c)$$

and $\forall\beta\in\mathcal{S}$ and $\forall i\in\mathcal{R}_{\beta(t_k)}$

$$\Pi_\beta^i = \begin{pmatrix} \pi_{11}^i(\beta) & \pi_{12}^i(\beta) & \pi_{13}^i(\beta) \\ * & \pi_{22}^i(\beta) & \pi_{23}^i(\beta) \\ * & * & \pi_{33}^i(\beta) \end{pmatrix} < 0 \quad (5)$$

$$\Psi_{\beta j}^i = \begin{pmatrix} T_{\beta 11}^i & T_{\beta 12}^i & X_\beta^i \\ * & T_{\beta 22}^i & Y_\beta^i \\ * & * & e^{-\alpha_\beta d_j} R_{j\beta}^i \end{pmatrix} \ge 0 \quad (6)$$

where

$$\pi_{11}^i(\beta) = P_\beta^i A_0(\beta) + A_0{}^T(\beta)P_\beta^i + \sum_{j=1}^m\left(Q_{j\beta}^i + d_j T_{\beta 11}^i\right) + \alpha_\beta P_\beta^i + m\left(X_\beta^i + X_\beta^{i\,T}\right),$$

$$\pi_{12}^i(\beta) = \begin{pmatrix} P_\beta^i A_1(\beta) + Y_\beta^{i\,T} - X_\beta^i + d_1 T_{\beta 12}^i \\ \vdots \\ P_\beta^i A_m(\beta) + Y_\beta^{i\,T} - X_\beta^i + d_m T_{\beta 12}^i \end{pmatrix}^T,$$

$$\pi_{13}^i(\beta) = \begin{pmatrix} d_1 A_1(\beta) R_{1\beta}^i & \cdots & d_m A_m(\beta) R_{m\beta}^i \end{pmatrix},$$

$$\pi_{22}^i(\beta) = -\mathrm{diag}\left\{e^{-\alpha_\beta d_1}(1-\sigma_1)Q_{1\beta}^i + Y_\beta^i + Y_\beta^{i\,T} - d_1 T_{\beta 22}^i, \dots, e^{-\alpha_\beta d_m}(1-\sigma_m)Q_{m\beta}^i + Y_\beta^i + Y_\beta^{i\,T} - d_m T_{\beta 22}^i\right\},$$

$$\pi_{23}^i(\beta) = \begin{pmatrix} d_1 A_1{}^T(\beta)R_{1\beta}^i & \cdots & d_m A_1{}^T(\beta) R_{m\beta}^i \\ \vdots & \ddots & \vdots \\ d_1 A_m{}^T(\beta) R_{1\beta}^i & \cdots & d_m A_m{}^T(\beta) R_{m\beta}^i \end{pmatrix},$$

$$\pi_{33}^i(\beta) = -\mathrm{diag}\left\{d_1 R_{1\beta}^i, \dots, d_m R_{m\beta}^i\right\},$$

then the switched system (1) is exponentially stable for any MDADT switching signals satisfying (3).

Proof: Consider the multiple discontinuous Lyapunov functional candidate for the switched system (1) with the following form $\forall\, t\in L_{\beta(t)}^i, i\in\mathcal{R}_{\beta(t)}, \beta\in\mathcal{S}$:

$$V_\beta{}^i(x(t)) = V_{\beta 1}^i(x(t)) + V_{\beta 2}^i(x(t)) + V_{\beta 3}^i(x(t)) \quad (7)$$

$$V_{\beta 1}^i(x(t)) = x^T(t)P_\beta^i x(t)$$

$$V_{\beta 2}^i(x(t)) = \sum_{j=1}^m \int_{t-d_j(t)}^t e^{\alpha_\beta(s-t)} x^T(s) Q_{j\beta}^i x(s)\,ds$$

$$V_{\beta 3}^i(x(t)) = \sum_{j=1}^m \int_{t-d_j}^t e^{\alpha_\beta(s-t)}\left(s-t+d_j\right)\dot{x}^T(s) R_{j\beta}^i \dot{x}(s)\,ds,$$

where $\alpha_\beta > 0, \beta\in\mathcal{G}$ and $\alpha_\beta < 0, \beta\in\mathcal{U}$ must be determined, and matrices $P_\beta^i > 0$, $Q_{j\beta}^i > 0$, $R_{j\beta}^i > 0$ are to be obtained. Then, for a fixed β, we have

$$\dot{V}_{\beta 1}^i(x(t)) = 2x^T(t)P_\beta^i\dot{x}(t) = 2x^T(t)P_\beta^i\left[A_0(\beta)x(t) + \sum_{j=1}^m A_j(\beta)x\left(t-d_j(t)\right)\right] \quad (8)$$

$$\begin{aligned}\dot{V}_{\beta 2}^i(x(t)) &= -\alpha_\beta V_{\beta 2}^i(x(t)) - \sum_{j=1}^m\left(1-\dot{d}_j(t)\right)e^{-\alpha_\beta d_j(t)}x\left(t-d_j(t)\right)^T Q_{j\beta}^i\, x\left(t-d_j(t)\right) + \sum_{j=1}^m x^T(t)Q_{j\beta}^i x(t) \\ &\le -\alpha_\beta\sum_{j=1}^m\int_{t-d_j(t)}^t e^{\alpha_\beta(s-t)}x^T(s)Q_{j\beta}^i x(s)\,ds - \sum_{j=1}^m(1-\sigma_j)e^{-\alpha_\beta d_j}x\left(t-d_j(t)\right)^T Q_{j\beta}^i\, x\left(t-d_j(t)\right) + \sum_{j=1}^m x^T(t)Q_{j\beta}^i x(t)\end{aligned} \quad (9)$$

$$\begin{aligned}\dot{V}_{\beta 3}^i(x(t)) &= \sum_{j=1}^m d_j\dot{x}^T(s)R_{j\beta}^i\dot{x}(s) - \alpha_\beta\sum_{j=1}^m\int_{t-d_j}^t e^{\alpha_\beta(s-t)}\left(s-t+d_j\right)\dot{x}^T(s)R_{j\beta}^i\dot{x}(s)\,ds - \sum_{j=1}^m\int_{t-d_j}^t e^{\alpha_\beta(s-t)}\dot{x}^T(s)R_{j\beta}^i\dot{x}(s)\,ds \\ &\le \sum_{j=1}^m d_j\left[A_0(\beta)x(t) + \sum_{j=1}^m A_j(\beta)x\left(t-d_j(t)\right)\right]^T R_{j\beta}^i\left[A_0(\beta)x(t) + \sum_{j=1}^m A_j(\beta)x\left(t-d_j(t)\right)\right]\end{aligned}$$

$$-\alpha_\beta\sum_{j=1}^{m}\int_{t-d_j}^{t}e^{\alpha_\beta(s-t)}(s-t+d_j)\dot{x}^T(s)R_{j\beta}^i\dot{x}(s)ds$$
$$-\sum_{j=1}^{m}e^{-\alpha_\beta d_j}\int_{t-d_j(t)}^{t}\dot{x}^T(s)R_{j\beta}^i\dot{x}(s)ds. \quad (10)$$

We know that $x(t)-x\left(t-d_j(t)\right)=\int_{t-d_j(t)}^{t}\dot{x}(s)ds$ for $1\le j\le m$, then, for any matrices X_β^i and Y_β^i with appropriate dimension, $\forall i\in\mathcal{R}_{\beta(t)},\beta\in\mathcal{S}$ we have

$$\sum_{j=1}^{m}2\left[x^T(t)X_\beta^i-x\left(t-d_j(t)\right)^TY_\beta^i\right]\left[x(t)-x\left(t-d_j(t)\right)-\int_{t-d_j(t)}^{t}\dot{x}(s)ds\right]=0. \quad (11)$$

For any matrix $T_\beta^i=\begin{pmatrix}T_{\beta11}^i & T_{\beta12}^i\\ T_{\beta21}^i & T_{\beta22}^i\end{pmatrix}\ge 0$, it yields

$$\sum_{j=1}^{m}d_j\begin{bmatrix}x^T(t)\\ x^T\left(t-d_j(t)\right)\end{bmatrix}^TT_\beta^i\begin{bmatrix}x^T(t)\\ x^T\left(t-d_j(t)\right)\end{bmatrix}-\sum_{j=1}^{m}\int_{t-d_j(t)}^{t}\begin{bmatrix}x^T(t)\\ x^T\left(t-d_j(t)\right)\end{bmatrix}^TT_\beta^i\begin{bmatrix}x^T(t)\\ x^T\left(t-d_j(t)\right)\end{bmatrix}ds\ge 0. \quad (12)$$

Regarding (8) to (12), we have

$$\dot{V}_\beta{}^i(x(t))+\alpha_\beta V_\beta{}^i(x(t))\le \varphi^T(t)\Pi_\beta^i\varphi(t)-\sum_{j=1}^{m}\int_{t-d_j(t)}^{t}\omega_j{}^T(t,s)\Psi_{\beta j}^i\omega_j(t,s) \quad (13)$$

where

$\varphi(t)=\left[x^T(t)\quad x^T(t-d_1(t))\quad\cdots\quad x^T(t-d_m(t))\right]^T$,

$\omega_j(t,s)=\left[x^T(t)\quad x\left(t-d_j(t)\right)\quad \dot{x}^T(s)\right]^T$, and

$$\widetilde{\Pi}_\beta^i=\begin{bmatrix}\pi_{11}^i(\beta) & \pi_{12}^i(\beta)\\ * & \pi_{22}^i(\beta)\end{bmatrix}+\sum_{j=1}^{m}d_j\begin{bmatrix}A_0(\beta)\\ A_1(\beta)\\ \vdots\\ A_\mathrm{m}(\beta)\end{bmatrix}R_{j\beta}^i[A_0(\beta)\quad A_1(\beta)\quad\cdots\quad A_\mathrm{m}(\beta)].$$

By Schur complement for (5) we can get $\widetilde{\Pi}_\beta^i\le 0$. Consequently, due to $\Psi_{\beta j}^i\ge 0$, one has

$$\dot{V}_\beta{}^i(x(t))+\alpha_\beta V_\beta{}^i(x(t))\le 0\quad \forall\beta\in\mathcal{S},i\in\mathcal{R}_{\beta(t)}. \quad (14)$$

According to (4.a) and (7), $\forall\,\beta\in\mathcal{S},i\ne 0$ we have

$$V_{\beta1}^i\left(x\left(t_k+J_\beta^i\right)\right)-\eta_\beta V_{\beta1}^{i-1}\left(x\left(t_k+J_\beta^i\right)\right)\le x^T(t)\left[P_\beta^i-\eta_\beta P_\beta^{i-1}\right]x(t)\le 0$$

$$V_{\beta2}^i\left(x\left(t_k+J_\beta^i\right)\right)-\eta_\beta V_{\beta2}^{i-1}\left(x\left(t_k+J_\beta^i\right)\right)\le\sum_{j=1}^{m}\int_{t-d_j(t)}^{t}e^{\alpha_\beta(s-t)}x^T(s)\left[Q_{j\beta}^i-\eta_\beta Q_{j\beta}^{i-1}\right]x(s)ds\le 0$$

$$V_{\beta3}^i\left(x\left(t_k+J_\beta^i\right)\right)-\eta_\beta V_{\beta3}^{i-1}\left(x\left(t_k+J_\beta^i\right)\right)\le\sum_{j=1}^{m}\int_{t-d_j}^{t}e^{\alpha_\beta(s-t)}(s-t+d_j)\dot{x}^T(s)\left[R_{j\beta}^i-\eta_\beta R_{j\beta}^{i-1}\right]\dot{x}(s)ds\le 0$$

therefore

$$\left[V_{\beta1}^i\left(x\left(t_k+J_\beta^i\right)\right)+V_{\beta2}^i\left(x\left(t_k+J_\beta^i\right)\right)+V_{\beta3}^i\left(x\left(t_k+J_\beta^i\right)\right)\right]-\eta_\beta\left[V_{\beta1}^{i-1}\left(x\left(t_k+J_\beta^i\right)\right)+V_{\beta2}^{i-1}\left(x\left(t_k+J_\beta^i\right)\right)+V_{\beta3}^{i-1}\left(x\left(t_k+J_\beta^i\right)\right)\right]\le 0$$

which yields $\forall\,\beta\in\mathcal{S},i\ne 0$

$$V_\beta{}^i\left(x\left(t_k+J_\beta^i\right)\right)-\eta_\beta V_\beta{}^{i-1}\left(x\left(t_k+J_\beta^i\right)\right)\le 0. \quad (15)$$

According to (4.b) and (7), $\forall(\beta,\gamma)\in\mathcal{G}\times\mathcal{S},\beta\ne\gamma$ we obtain

$$V_{\beta1}^0(x(t))-\mu_\beta V_{\gamma1}^{G_\gamma-1}(x(t))=x^T(t)\left[P_\beta^0-\mu_\beta P_\gamma^{G_\gamma-1}\right]x(t)\le 0$$

$$V_{\beta2}^0(x(t))-\mu_\beta V_{\gamma2}^{G_\gamma-1}(x(t))=\sum_{j=1}^{m}\int_{t-d_j(t)}^{t}e^{\alpha_\beta(s-t)}x^T(s)\left[Q_{j\beta}^0-\mu_\beta Q_{j\gamma}^{G_\gamma-1}\right]x(s)ds\le 0$$

$$V_{\beta3}^0(x(t))-\mu_\beta V_{\gamma3}^{G_\gamma-1}(x(t))=\sum_{j=1}^{m}\int_{t-d_j}^{t}e^{\alpha_\beta(s-t)}(s-t+d_j)\dot{x}^T(s)\left[R_{j\beta}^0-\mu_\beta R_{j\gamma}^{G_\gamma-1}\right]\dot{x}(s)ds\le 0$$

so,

$$\left[V_{\beta1}^0(x(t))+V_{\beta2}^0(x(t))+V_{\beta3}^0(x(t))\right]-\mu_\beta\left[V_{\gamma1}^{G_\gamma-1}(x(t))+V_{\gamma2}^{G_\gamma-1}(x(t))+V_{\gamma3}^{G_\gamma-1}(x(t))\right]\le 0$$

which yields $\forall(\beta,\gamma)\in\mathcal{G}\times\mathcal{S},\beta\ne\gamma$

$$V_\beta{}^0(x(t))-\mu_\beta V_\gamma{}^{G_\gamma-1}(x(t))\le 0. \quad (16)$$

According to (4.c) and (7), $\forall(\beta,\gamma)\in\mathcal{G}\times\mathcal{U}$ we have

$$V_{\gamma1}^0(x(t))-\mu_\gamma V_{\beta1}^{G_\beta-1}(x(t))=x^T(t)\left[P_\gamma^0-\mu_\gamma P_\beta^{G_\beta-1}\right]x(t)\le 0$$

$$V_{\gamma2}^0(x(t))-\mu_\gamma V_{\beta2}^{G_\beta-1}(x(t))=\sum_{j=1}^{m}\int_{t-d_j(t)}^{t}e^{\alpha_\beta(s-t)}x^T(s)\left[Q_{j\gamma}^0-\mu_\gamma Q_{j\beta}^{G_\beta-1}\right]x(s)ds\le 0$$

$$V_{\gamma 3}^{0}\big(x(t)\big)-\mu_\gamma V_{\beta 3}^{G_\beta -1}\big(x(t)\big) = \sum_{j=1}^{m}\int_{t-d_j}^{t} e^{\alpha_\beta (s-t)}\big(s-t + d_j\big)\dot{x}^T(s)\left[R_{j\gamma}^{0}-\mu_\gamma R_{j\beta}^{G_\beta -1}\right]\dot{x}(s)ds \leq 0$$

yields

$$\left[V_{\gamma 1}^{0}\big(x(t)\big)+V_{\gamma 2}^{0}\big(x(t)\big)+V_{\gamma 3}^{0}\big(x(t)\big)\right] - \mu_\gamma\left[V_{\beta 1}^{G_\beta -1}\big(x(t)\big)+V_{\beta 2}^{G_\beta -1}\big(x(t)\big) + V_{\beta 3}^{G_\beta -1}\big(x(t)\big)\right]\leq 0$$

and implies $\forall(\beta,\gamma)\in\mathcal{G}\times\mathcal{U}$

$$V_\gamma{}^{0}\big(x(t)\big)-\mu_\gamma V_\beta{}^{G_\beta -1}\big(x(t)\big)\leq 0. \qquad (17)$$

By considering (14)-(17), according to Theorem 1 in [12], if τ_β^a satisfies (3), on the time interval $[0;T]$, we have

$$V_{\beta(T^-)}(T^-)\leq exp\left\{\sum_{\beta\in\mathcal{G}}\big(N_{0\beta}\, ln\,\mu_\beta(\eta_\beta)^{G_\beta -1}\big) + \sum_{\beta\in\mathcal{U}}\big(N_{0\beta}\, ln\,\mu_\beta(\eta_\beta)^{G_\beta -1}\big)\right\} \times e^{\max_{\beta\in\mathcal{S}}\left\{\left(\frac{ln\,\mu_\beta(\eta_\beta)^{G_\beta -1})}{\tau_\beta^a}-\alpha_\beta\right)T_\beta(T,0)\right\}} \times(\eta_{\beta(0)})^{G_{\beta(0)}-1}V_{\beta(0)}^{0}\big(x(0)\big),$$

which $N_{0\beta}$ is a constant number. So, one can conclude that for any MDADT switching signal satisfying (3), $V_{\beta(T^-)}\big(x(T)\big)$ converges to zero as $T\to\infty$.

Finally, by Definition 1 and the first equation of Lemma 1, we can get that switched system (3) is exponentially stable, and the exponential decay rate is equal to $\max_{\beta\in\mathcal{S}}\left\{\frac{\ln\mu_\beta(\eta_\beta)^{G_\beta -1})}{\tau_\beta^a}-\alpha_\beta\right\}$. □

Remark 1: As mentioned before, in our switching mechanisms in Theorem 1, using the MDADT method, we design fast switching and slow switching for unstable and stable subsystems, respectively. This not only gives lower bounds that stable subsystems should dwell on but also provides the upper bounds that the MDADT of unstable modes cannot exceed. Such a switching strategy enables us to easily balance the dwell time between unstable and stable subsystems in a mode-dependent manner. This switching strategy cannot be applied to some other time-dependent switching signals like DT, ADT, due to the fact that they are not set in a mode-dependent manner. On the other hand, it must be said that the MDLF in mentioned Theorem, is only a piecewise continuous function during the dwell time on each mode, unlike some common MLFs for switched systems, which requires that the Lyapunov function for each mode is continuously differentiable during the running time. Based on such a Lyapunov functional, tighter bounds on dwell time can be obtained, which undoubtedly enhances the application flexibility in practice. In MDLF if we choose $G_\beta = 1$, which means that there is only one continuous Lyapunov function between two consecutive switching, then the MDLF directly reduces MLF. Thus, MLF is a special case of MDLF. One of the challenging issues of the MDLF is that all subsystems cannot be unstable without considering the case that when switched systems composed of stable and unstable subsystems, which may bring some limitations in actual operations.

Remark 2: In Theorem 1, computable sufficient conditions formulated in the form of LMIs. When solving these LMIs, the parameters α_β, G_β, η_β, and μ_β should be given in advance. To search tighter bounds on MDADT, the following procedures are introduced in [12]. To choose the corresponding parameters for the LMIs: First, noticing that G_β do not affect the feasibility of the LMIs, so if the computer computation ability allows, G_β can be chosen big enough, but otherwise, G_β can be selected bigger for those modes needing tighter bounds on dwell time and smaller for the other modes to prevent the complexity of computation. Second, the convergence rate (or divergence rate) α_β, can be estimated based on the eigenvalues of each subsystem. Third, setting the step lengths and the initial values of $\eta_\beta$and μ_β, we can apply a two-layer loop program under the condition $(\eta_\beta)^{G_\beta -1}\mu_\beta > 1$ to solve the LMI conditions. Then, tighter bounds on MDADT can be identified among the obtained feasible solutions. It should be pointed out that from the energy view, the parameter η_β also influence the decay rate α_β of the MDLF. So that smaller η_β can cause the infeasibility problem of the conditions. Hence, G_β and η_β should be carefully designed depending on the practical situations. However, it is worth noting that obtaining minimal MDADT is still an unsolved problem.

IV.NUMERICAL EXAMPLES

In this section, we present examples illustrating how our theoretical results can be applied to switched systems with multiple time-varying delays.

Example 1: Consider a switched system with $m = 2$, $N = 2$, and system parameters as followings:

Subsystem 1 (Stable): $A_0(1)=\begin{pmatrix}0.2 & -1\\ 0.3 & -0.5\end{pmatrix}$, $A_1(1)=\begin{pmatrix}0.1 & 0.2\\ 0 & -0.3\end{pmatrix}$, $A_2(1)=\begin{pmatrix}-0.1 & 0\\ 0.1 & 0.4\end{pmatrix}$.

Subsystem 2 (Unstable): $A_0(2)=\begin{pmatrix}-0.3 & 1\\ 0 & 0.5\end{pmatrix}$, $A_1(2)=\begin{pmatrix}0.1 & 0.1\\ 0 & -0.2\end{pmatrix}$, $A_2(2)=\begin{pmatrix}-0.5 & 0\\ 0.2 & 0.2\end{pmatrix}$.

and $d_1(t) = 0.05 + 0.05\sin(t)$, $d_2(t) = 0.2 + 0.1\sin(t)$. The switched system under the random switching signal becomes unstable, as shown in Fig. 1.

Now, the stability of the switched system, under

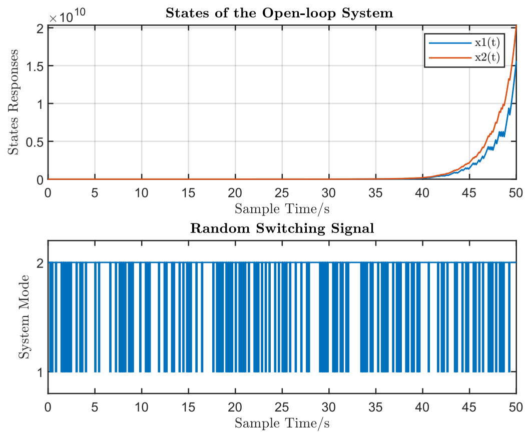

Fig. 1. States of the system with random switching signal.

designed MDADT switching, is checked. By choosing $\alpha_1 = 2.9$, $\alpha_2 = -3.2$, and $G_1 = G_2 = 3$ and $\mu_1 = 6$, $\mu_2 = 0.7$ and $\eta_1 = \eta_2 = 0.7$, LMIs Conditions (4) to (6) is feasible, and by using above parameters, we obtain MDADT switching signals, $\tau_1^{a*} = 0.945$, $\tau_2^{a*} = 0.551$. Then, by giving initial state condition $x(\theta) = (10 \quad -5)^T$, $\theta \in [-0.3,0]$, and the possible switching signals $\tau_1^a \geq \tau_1^{a*} = 0.95$, $\tau_2^a \leq \tau_2^{a*} = 0.5$, the resultant state responses and the phase trajectories of the system under designed switching signal, are shown in Fig. 2. From Fig. 2 it can be seen the system is stable under our designed MDADT switching signal, which shows the effectiveness of the results.

Example 2: Consider the switched delay system (1) as in [36], with $m = 2$, $N = 2$, and system parameters as followings:

Subsystem 1 (Unstable): $A_0(1) = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$,

$A_1(1) = \begin{pmatrix} 0.1 & -0.1 \\ 0.2 & 0.1 \end{pmatrix}$, $A_2(1) = \begin{pmatrix} -0.1 & 0.2 \\ -0.2 & 0.1 \end{pmatrix}$.

Subsystem 2 (Stable): $A_0(2) = \begin{pmatrix} -0.3 & 1 \\ 0 & 0.5 \end{pmatrix}$,

$A_1(2) = \begin{pmatrix} 0.1 & -0.1 \\ 0.2 & 0.1 \end{pmatrix}$, $A_2(2) = \begin{pmatrix} -0.1 & 0.2 \\ -0.2 & 0.1 \end{pmatrix}$.

We compare our results with the one in [36]. By using Theorem 1, and choosing $G_1 = G_2 = 3$, and considering TABLE I, which represents the design parameters, the LMIs conditions (4)-(6) is feasible. The comparison results are listed in TABLE II.

TABLE I
DESIGN PARAMETERS

Methods	$d_1 = d_2 = 0.5$, $\sigma_1, \sigma_2 \leq 0.1693$
Theorem 1 of [36]	$\mu = 1.053$ $\alpha = 0.05$
Theorem 1	$\mu_1 = 9.4, \mu_2 = 0.89$ $\eta_1 = \eta_2 = 0.9$ $\alpha_1 = 1, \alpha_2 = -1.1$

TABLE II
COMPARATIVE RESULTS OF τ_β^{a*}

Methods	$d_1 = d_2 = 0.5$, $\sigma_1, \sigma_2 \leq 0.1693$
Theorem 1 of [36]	τ^{a*} =*0.5164* $\tau_1^a \geq \tau^{a*} = 4.6, \tau_2^a \leq \tau^{a*} = 0.4$
Theorem 1	$\tau_1^{a*} = 2.03, \tau_2^{a*} = 0.298$ $\tau_1^a \geq \tau_1^{a*} = 2.05, \tau_2^a \leq \tau_2^{a*} = 0.29$

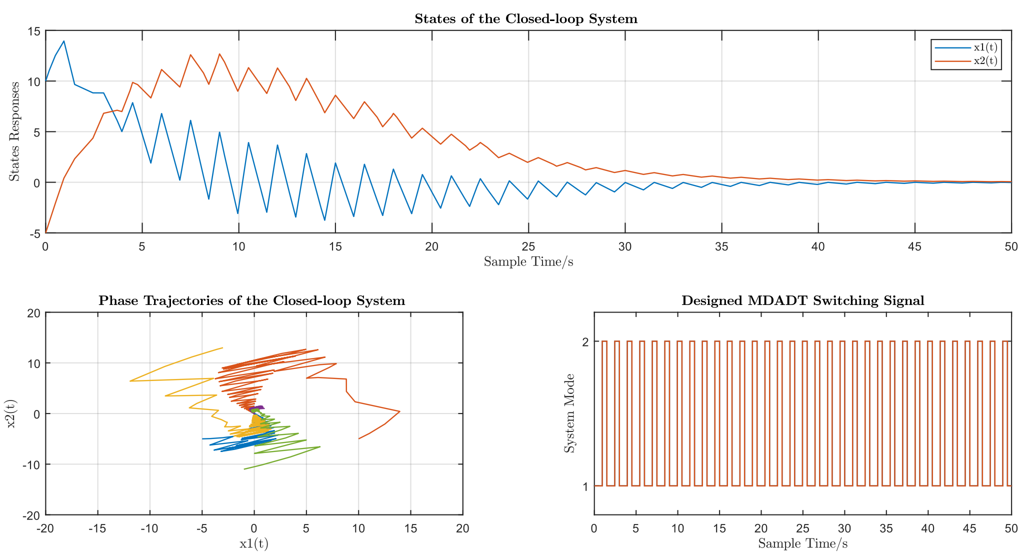

Fig. 2. States of the system for $x(\theta) = (10 \quad -5)^T$, and convergent phase trajectories for different initial conditions of the system with designed MDADT switching signal.

It is clear that the obtained results in this paper are less conservative than the results of [36], in determining the tighter dwell time band. Because according to Theory 1 of reference of [36], in a switching period of time that is $\tau_1^a + \tau_2^a = 5$ seconds, the dwell time of the unstable subsystem is 0.4 seconds and equal to 8% of the total switching time. But according to the proposed method in Theorem 1, in the same time period, the dwell time of the unstable subsystem is equal to 12.8% of the total switching time, which makes the tighter dwell time band. By considering TABLE II, we can obtain the state response with initial state condition $x(\theta) = (10 \quad 20)^T, \theta \in [-0.6693,0]$ and the phase trajectories of the system, as shown in Fig. 3. We can see that the switched system is stable under designed MDADT switching signal.

V.CONCLUSION

In this paper, stability analysis was investigated for a class of switched systems with multiple time-varying delays. By using, the MDLF approach and using MDADT switching regime, sufficient conditions in the form of LMIs have been proposed to guarantee the exponential stability of the switched system with the existence of unstable subsystems, which offers a tighter dwell time-bound with less conservativeness. Finally, two examples were given to illustrate the effectiveness of the obtained theoretical results, and to clarify the differences with other works the proposed method was compared with the existing methods. The comparison showed that a tighter bound of dwell time, can be achieved with the proposed me

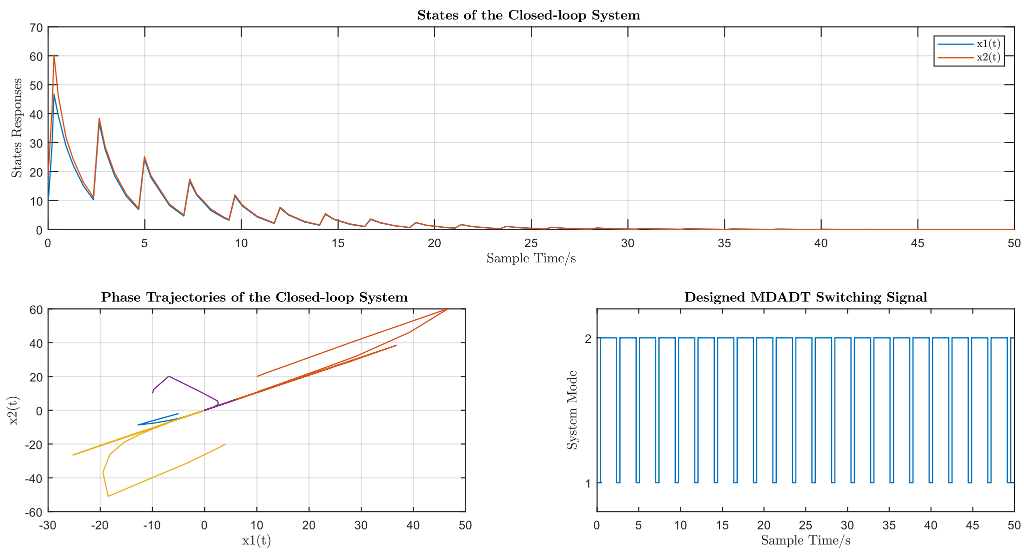

Fig. 3. States of the system for $x(\theta) = (10 \quad 20)^T$, and convergent phase trajectories for different initial conditions of the system with designed MDADT switching signal.

REFERENCES AND FOOTNOTES

[1] C. Briat, "Dwell-time stability and stabilization conditions for linear positive impulsive and switched systems," *Nonlinear Analysis: Hybrid Systems*, vol. 24, pp. 198-226, 2017.

[2] W. Ren and J. Xiong, "Stability analysis of impulsive switched time-delay systems with state-dependent impulses," *IEEE Transactions on Automatic Control*, vol. 64, no. 9, pp. 3928-3935, 2019.

[3] C. Zhu, X. Li, and J. Cao, "Finite-time H∞ dynamic output feedback control for nonlinear impulsive switched systems," *Nonlinear Analysis: Hybrid Systems*, vol. 39, p. 100975, 2021.

[4] M. Xue, Y. Tang, W. Ren, and F. Qian, "Practical output synchronization for asynchronously switched multi-agent systems with adaption to fast-switching perturbations," *Automatica*, vol. 116, p. 108917, 2020.

[5] X. Su, X. Liu, P. Shi, and Y.-D. Song, "Sliding mode control of hybrid switched systems via an event-triggered mechanism," *Automatica*, vol. 90, pp. 294-303, 2018.

[6] G. Zong, H. Ren, and H. R. Karimi, "Event-triggered communication and annular finite-time H∞ filtering for networked switched systems," *IEEE Transactions on Cybernetics*, vol. 51, no. 1, pp. 309-317, 2020.

[7] B. Niu, X. Zhao, X. Fan, and Y. Cheng, "A new control method for state-constrained nonlinear switched systems with application to chemical process,"

International Journal of Control, vol. 88, no. 9, pp. 1693-1701, 2015.

[8] L. Li, L. Song, T. Li, and J. Fu, "Event-triggered output regulation for networked flight control system based on an asynchronous switched system approach," *IEEE Transactions on Systems, Man, and Cybernetics: Systems,* 2020.

[9] X. Su, X. Liu, P. Shi, and R. Yang, "Sliding mode control of discrete-time switched systems with repeated scalar nonlinearities," *IEEE Transactions on Automatic Control,* vol. 62, no. 9, pp. 4604-4610, 2016.

[10] J. Wei, H. Zhi, K. Liu, and X. Mu, "Stability of mode-dependent linear switched singular systems with stable and unstable subsystems," *Journal of the Franklin Institute,* vol. 356, no. 5, pp. 3102-3114, 2019.

[11] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE control systems magazine,* vol. 19, no. 5, pp. 59-70, 1999.

[12] X. Zhao, P. Shi, Y. Yin, and S. K. Nguang, "New results on stability of slowly switched systems: A multiple discontinuous Lyapunov function approach," *IEEE Transactions on Automatic Control,* vol. 62, no. 7, pp. 3502-3509, 2016.

[13] D. Yang, X. Li, and J. Qiu, "Output tracking control of delayed switched systems via state-dependent switching and dynamic output feedback," *Nonlinear Analysis: Hybrid Systems,* vol. 32, pp. 294-305, 2019.

[14] W. Xiang and J. Xiao, "Stabilization of switched continuous-time systems with all modes unstable via dwell time switching," *Automatica,* vol. 50, no. 3, pp. 940-945, 2014.

[15] Y. Yin, G. Zong, and X. Zhao, "Improved stability criteria for switched positive linear systems with average dwell time switching," *Journal of the Franklin Institute,* vol. 354, no. 8, pp. 3472-3484, 2017.

[16] X. Zhao, L. Zhang, P. Shi, and M. Liu, "Stability and stabilization of switched linear systems with mode-dependent average dwell time," *IEEE Transactions on Automatic Control,* vol. 57, no. 7, pp. 1809-1815, 2011.

[17] Z. Fei, S. Shi, C. Zhao, and L. Wu, "Asynchronous control for 2-D switched systems with mode-dependent average dwell time," *Automatica,* vol. 79, pp. 198-206, 2017.

[18] B. Niu, P. Zhao, J.-D. Liu, H.-J. Ma, and Y.-J. Liu, "Global adaptive control of switched uncertain nonlinear systems: An improved MDADT method," *Automatica,* vol. 115, p. 108872, 2020.

[19] H. Hu, Y. Li, J. Liu, E. Tian, and X. Xie, "Fault estimation for delta operator switched systems with mode-dependent average dwell-time," *Journal of the Franklin Institute,* 2021.

[20] J. Hosseini, Z. Rahmani, and A. Ranjbar Noe, "Adaptive tracking control of switched linear systems with multiple disturbance observers," *Transactions of the Institute of Measurement and Control,* vol. 43, no. 16, pp. 3499-3518, 2021.

[21] F. Liao, J. Wu, and M. Tomizuka, "An improved delay-dependent stability criterion for linear uncertain systems with multiple time-varying delays," *International Journal of Control,* vol. 87, no. 4, pp. 861-873, 2014.

[22] K. S. Ko, W. I. Lee, P. Park, and D. K. Sung, "Delays-dependent region partitioning approach for stability criterion of linear systems with multiple time-varying delays," *Automatica,* vol. 87, pp. 389-394, 2018.

[23] Y. He, M. Wu, and J.-H. She, "Delay-dependent stability criteria for linear systems with multiple time delays," *IEE Proceedings-Control Theory and Applications,* vol. 153, no. 4, pp. 447-452, 2006.

[24] Z.-Y. Li, Q. Fan, L. Li, and Y. Wang, "Stability analysis of linear continuous-time delay-difference systems with multiple time-delays," *Journal of the Franklin Institute,* vol. 356, no. 18, pp. 11492-11519, 2019.

[25] F. S. de Oliveira and F. O. Souza, "Improved delay-dependent stability criteria for linear systems with multiple time-varying delays," *International Journal of Control,* vol. 94, no. 12, pp. 3415-3423, 2021.

[26] J. Tian, S. Zhong, and L. Xiong, "Delay-dependent absolute stability of Lurie control systems with multiple time-delays," *Applied mathematics and computation,* vol. 188, no. 1, pp. 379-384, 2007.

[27] M. S. Mahmoud, "Switched time-delay systems," in *Switched time-delay systems*: Springer, 2010, pp. 109-130.

[28] S. Pezeshki, M. A. Badamchizadeh, A. Rikhtehgar Ghiasi, and S. Ghaemi, "Stability analysis and robust stabilisation for a class of switched nonlinear systems with input delay," *International Journal of Control,* vol. 93, no. 10, pp. 2457-2468, 2020.

[29] J.-y. Zhai, B. Wang, and S.-m. Fei, "Tracking control for switched nonlinear systems with multiple time-varying delays," *Nonlinear Analysis: Hybrid Systems,* vol. 17, pp. 44-55, 2015.

[30] I. Zamani, M. Shafiee, and A. Ibeas, "Stability analysis of hybrid switched nonlinear singular time-delay systems with stable and unstable subsystems," *International Journal of Systems Science,* vol. 45, no. 5, pp. 1128-1144, 2014.

[31] I. Zamani and M. Shafiee, "On the stability issues of switched singular time-delay systems with slow switching based on average dwell-time," *International Journal of Robust and Nonlinear Control,* vol. 24, no. 4, pp. 595-624, 2014.

[32] I. Zamani, M. Shafiee, and A. Ibeas, "Switched nonlinear singular systems with time-delay: Stability analysis," *International Journal of Robust and Nonlinear Control,* vol. 25, no. 10, pp. 1497-1513, 2015.

[33] I. Zamani, M. Shafiee, and A. Ibeas, "On singular hybrid switched and impulsive systems," *International Journal of Robust and Nonlinear Control,* vol. 28, no. 2, pp. 437-465, 2018.

[34] M. M. Saberi and I. Zamani, "Stability and stabilisation of switched time-varying delay systems: a multiple discontinuous Lyapunov function approach," *International Journal of Systems Science,* vol. 51, no. 13, pp. 2378-2409, 2020.

[35] J. K. Hale and S. M. V. Lunel, *Introduction to functional differential equations*. Springer Science & Business Media, 2013.

[36] L. Wang and C. Shao, "Robust stability analysis of switched uncertain systems with multiple time-varying delays and actuator failures," *Journal of Control Theory and Applications,* vol. 9, no. 2, pp. 267-272, 2011.